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**Beyond risk-based portfolios:
Balancing performance and risk
contributions in asset allocation**

Executive Summary

Optimal investing only takes the risks that are worthwhile. The first generation of risk-based portfolios completely ignores the expected return dimension when defining portfolio weights. At Finvex, we therefore pride ourselves on a thorough screening of the investment universe to obtain a selection of sustainable assets with an attractive return/risk profile. In the absence of such screening, a naive implementation of risk-based portfolios may lead to inefficiencies in terms of overweighting assets with close to zero or even negative expected return contributions.

In this paper, researchers at the Finvex Quantitative Strategies team introduce a methodology to enhance risk-based portfolios by including views on expected returns. The motivation is that the weight of an asset can be increased incrementally as long as the marginal gain in performance exceeds sufficiently the marginal cost in terms of the increased risk. The proposed solution consists of first identifying the positions for which the performance and risk contributions are not aligned and then modifying the risk-based portfolio weights such that they strike a balance between return and risk contributions. The analysis is performed by considering various risk measures, including downside risk measures, taking the non-normality of the return distribution into account.

The authors illustrate the methodology for an asset allocation example where they show that especially the approach of equal-weighting benefits from the enhancement of introducing views on expected returns.

I wish you a pleasant reading and please don't hesitate to contact us for any further inquiries.

Kind regards,

Stefan Hartmann,

Head of Quantitative Research, Finvex

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Beyond Risk-Based Portfolios: Balancing Performance and Risk Contributions in Asset Allocation[☆]

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Abstract

In a risk-based portfolio, there is no explicit control for the performance per unit of risk taken. We propose a framework to evaluate the balance between risk and performance at both the portfolio and component level, and to tilt the risk-based portfolio weights towards a state in which the performance and risk contributions are aligned. The key innovation is the *Performance/Risk Contribution Concentration* (PRCC) measure, which is designed to be minimal when, for all portfolio components, the performance and risk contributions are perfectly aligned. We investigate the theoretical properties of this measure and show its usefulness to obtain the PRCC modified risk-based portfolio weights, that avoid excesses in terms of deviations between the performance and risk contributions of the portfolio components, while still being close to the benchmark risk-based portfolio in terms of weights and relative performance.

Keywords: Asset allocation, Performance/Risk Contribution, Target relative performance portfolio

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1. Introduction

The percentage capital allocation is well known to be a bad advisor on the percentage risk allocation in multi-asset class portfolios. In a typical 60/40 US equities-bond portfolio, the equity part is often responsible for more than 90% of the total portfolio's volatility (Qian, 2005). One solution is to let portfolio weights be indirectly determined by a target constraint on the percentage volatility contributions. A special case is the risk parity or equal-risk-contribution portfolio, seeking for portfolios in which all components contribute equally to the portfolio's volatility (see, *e.g.*, Qian (2005), Maillard et al. (2010) and Bai et al. (2016)).

Boudt et al. (2012) and Roncalli (2015) generalize the approach to risk allocation based on downside risk measures of the type $\mathcal{R}_p \equiv -\mu_p + c_p \sigma_p$, with μ_p and σ_p the portfolio expected return and volatility, and c_p a multiple that may depend on the portfolio return distribution. For such downside risk measures, setting a target value on the risk contribution implies finding a balance between the expected return contribution and the portfolio volatility contribution. This objective of finding a balance between a *marginal revenue*-type measure and a *marginal cost*-type measure is intuitive from a profit-maximizing perspective, as mentioned by Lee (2011). In fact, the maximum Sharpe ratio portfolio is such that the excess return contribution of each asset is proportional to the volatility contribution of that asset, with the value of the multiplier being equal to the value of the maximum Sharpe ratio.

Because of the non-normality of financial returns, the portfolio mean and volatility are not sufficient to describe the preference of most investors. As shown by Scott and Horvath (1980), investors tend to have positive preferences for odd moments (*i.e.*, mean and skewness) and aversion to even moments (*i.e.*, variance and kurtosis). It follows that balancing the excess return contributions of the portfolio positions with their volatility contribution may not be optimal when the returns have a non-normal distribution. A potentially better approach is thus to balance the expected (excess) return contribution with the contribution to a portfolio downside risk measure, taking the non-normality of the return distribution into account.

In this paper, we introduce a flexible framework to evaluate and optimize the balance between components' performance and risk contributions, where the performance measure (denoted by \mathcal{P}_p) and risk measure (denoted by \mathcal{R}_p) can be any measure, as long as they are first-order homogeneous functions of the portfolio weights, such that they can be decomposed into performance and risk contributions using Euler's theorem. This is clearly the case for the portfolio mean (excess) return, and for the portfolio volatility, when estimated using the classical sample-based estimator. In our study, we consider also four estimators for the portfolio Value-at-Risk (VaR) and Expected Shortfall (ES) that are first-degree homogeneous functions of the portfolio weights: the parametric approaches of assuming a Gaussian or a Student- t distribution, the semi-parametric approach based on the Cornish-Fisher approximation, and the non-parametric technique using kernel estimators.

For the evaluation of the balance between the performance and risk contributions, we propose the Performance/Risk Contribution Concentration (PRCC) metric. This measure is designed to be minimal when, for all portfolio components, the performance and risk contributions are perfectly aligned. We show its usefulness as an *ex-post* diagnostic tool to characterize the portfolio's bets in terms of performance contributions. More precisely, we define a bet when the ratio between the portfolio component's performance contribution compared with its risk contribution deviates from the ratio between the aggregate portfolio performance relative to the aggregate portfolio risk. The latter is denoted as $\tau_p \equiv \mathcal{P}_p/\mathcal{R}_p$ and henceforth used as a measure of the portfolio's relative performance.

The potential mismatch between the component performance and risk contribution of a portfolio is especially a concern in the analysis of a risk-based portfolio, which, by the definition of Lee (2011), is a portfolio for which the weights are determined without making use of a return forecast. A typical example is the equally-weighted portfolio, for which Kritzman et al. (2010, page 31) criticize the absence of optimization as follows: "If we have at least some information on the expected returns, riskiness, and diversification properties of the assets, why should we not

expect optimization to improve on a naively diversified portfolio?” On the other hand, Ardia and Boudt (2015) show that risk-based portfolios can have the maximum Sharpe ratio properties under specific conditions on the expected return. We illustrate in this paper that a necessary condition for a portfolio to be the maximum Sharpe ratio portfolio is that its PRCC equals zero (when the PRCC is computed using volatility as the risk measure). For this reason, we recommend to adjust the weights of risk-based portfolios such that their PRCC value are closer to zero. A further advantage of low values for the PRCC is that it implies robustness to estimation errors in the portfolio weights.

In addition to its interpretation as a diagnostic tool, we recommend using the proposed PRCC to go beyond risk-based portfolios in order to adjust the risk-based portfolio weights, such that they achieve a better balance between the performance and risk contributions at the individual component level. The adjustment is limited because of an upper-bound constraint on the mean squared distance between the PRCC modified weights and the original risk-based portfolio weights. This bound constraint ensures that the optimized weights can still be interpreted in connection to the traditional risk-based portfolio. We further impose that the optimized portfolio needs to have the same relative performance as the risk-based portfolio such that the PRCC is well defined. The proposed framework of considering performance and risk allocation jointly is an alternative to the traditional mean-variance optimization of Markowitz (1952).

We illustrate this framework of building PRCC modified risk-based portfolios in the real-life asset allocation problem of finding the optimal mix across investments in developed markets’ equity, emerging markets’ equity, US government bond, corporate bonds, real estate and gold over the period 1988-2015. Our out-of-sample analysis shows that, when the reference portfolio is the equally-weighted, equal-risk-contribution and maximum diversification portfolio, the PRCC is relatively high, and optimizing the PRCC under the constraint of equal relative performance and a maximum tracking error in terms of the portfolio weights, leads to a substantial increase in both the portfolio’s absolute and relative performance.

The remainder of the paper is organized as follows. In Section 2 we define the PRCC measure and show that it can be interpreted as a measure of robustness of the portfolio performance and risk with respect to weight perturbations. In Section 3 we study its properties in the case of a volatility-based risk measure for risk-based portfolios. We also show how it can be implemented in case of a downside risk objective. In Section 4 we introduce a portfolio optimization framework in which the investor minimizes the PRCC under a tracking error constraint with respect to the weights of a reference portfolio and ensures that the optimized portfolio has the same relative performance as the benchmark portfolio. In Section 5 we illustrate the use of the PRCC as a diagnostic and optimization criterion in a real-life asset allocation problem. Our main findings are summarized in Section 6.

2. Measuring the alignment of Component Performance/Risk Contributions

2.1. General framework

We consider a portfolio invested in N assets with weight vector $\mathbf{w} \equiv (w_1, \dots, w_N)'$.¹ We assume to have a measure for the performance of the portfolio, denoted by $\mathcal{P}_p(\mathbf{w})$, and a measure for the portfolio risk, denoted by $\mathcal{R}_p(\mathbf{w})$. As mentioned in Caporin et al. (2014), it is common to evaluate the portfolio's relative performance using ratios expressing the reward per unit of risk:

$$\tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w}) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})}. \quad (1)$$

As will be discussed later, many of the often used performance and risk measures have the property of being first-degree homogeneous functions of the portfolio weights. This means that if the portfolio weights are multiplied by a strictly positive scalar k , then the performance and risk measures are multiplied by k (i.e., $\mathcal{P}_p(k\mathbf{w}) = k\mathcal{P}_p(\mathbf{w})$ and $\mathcal{R}_p(k\mathbf{w}) = k\mathcal{R}_p(\mathbf{w})$ for $k > 0$). Examples

¹We present our framework to align the performance and risk contributions at the component level. An alternative is to do this at the factor level. We refer the interested reader to Boudt and Peeters (2013) and Roncalli and Weisang (2016) for an introduction to the methodology to compute and optimize the risk contributions at the factor level.

include all types of excess portfolio returns as performance measures and portfolio volatility, VaR and ES under the assumption of elliptically symmetric return distributions and modified downside risk measures under the Cornish-Fisher expansion as risk measures. From Euler's homogeneous function theorem, it follows that first-degree homogeneity is a useful property for performance and risk measures, as it implies the following aggregation results:

$$\begin{aligned}\mathcal{P}_p(\mathbf{w}) &= \sum_{i=1}^N w_i \partial_i \mathcal{P}_p(\mathbf{w}) \\ \mathcal{R}_p(\mathbf{w}) &= \sum_{i=1}^N w_i \partial_i \mathcal{R}_p(\mathbf{w}),\end{aligned}$$

where we denote the partial derivative $\frac{\partial}{\partial w_i}$ by ∂_i . In the literature on performance and risk budgeting, the term:

$$C_i^{\mathcal{P}}(\mathbf{w}) \equiv w_i \partial_i \mathcal{P}_p(\mathbf{w}), \quad (2)$$

is called the *component contribution* to the portfolio performance, and:

$$C_i^{\mathcal{R}}(\mathbf{w}) \equiv w_i \partial_i \mathcal{R}_p(\mathbf{w}), \quad (3)$$

is the *component risk contribution* (see, e.g., Boudt et al., 2008).

From the definition of $\tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w})$ as the relative performance measure in (1), it follows that the aggregate balance between the performance and risk contribution is:

$$\mathcal{P}_p(\mathbf{w}) = \tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w}) \mathcal{R}_p(\mathbf{w}).$$

But how is this balanced between the portfolio performance and risk distributed across the different positions? To answer this question, we need to investigate the balance in the performance and risk

contribution at the component level. To do so, let us define the *Component Performance/Risk Contribution* of asset i as:

$$\text{CPRC}_i(\mathbf{w}) \equiv C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})C_i^{\mathcal{R}}(\mathbf{w}).$$

Due to the first-degree homogeneity property and Euler's theorem, we have that the sum of all component performance/risk contributions is always zero:

$$\sum_{i=1}^N \text{CPRC}_i(\mathbf{w}) = 0. \quad (4)$$

As an aggregate measure of the dispersion in balance between the performance and risk contributions, we propose to use the following *Performance/Risk Contribution Concentration* measure for portfolios invested in multiple assets²:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{2N^2} \sum_{i=1}^N \sum_{j=1}^N \left\{ \left[C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})C_i^{\mathcal{R}}(\mathbf{w}) \right] - \left[C_j^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})C_j^{\mathcal{R}}(\mathbf{w}) \right] \right\}^2, \quad (5)$$

where we scale with $1/(2N^2)$ because of the property that the CPRC_i 's add up to zero in (4), which implies that many terms in the $\text{PRCC}(\mathbf{w})$ cancel out. In fact, as we show in Appendix A, it is equivalent to define the PRCC as the average squared value of the CPRC_i 's:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{N} \sum_{i=1}^N \left[C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})C_i^{\mathcal{R}}(\mathbf{w}) \right]^2 = \frac{1}{N} \sum_{i=1}^N [\text{CPRC}_i(\mathbf{w})]^2. \quad (6)$$

In practice, we use the computationally simpler expression (6) to calculate the PRCC, but for ease of interpretation, we refer to (5) as the main definition of the PRCC. Indeed, the most natural

²In the trivial case of a portfolio fully invested in a single asset, there is of course no dispersion and the PRCC is zero. Throughout the paper, we assume the portfolios to be invested in at least two assets.

interpretation of the PRCC measure is that it measures the concentration in the mismatch between performance and risk contributions of financial portfolios. The higher the PRCC is, the more concentrated the portfolio is in terms of positions where the performance contribution diverges from the risk contribution scaled by the portfolio's relative performance ratio.

2.2. The PRCC as a measure of robustness to perturbations in the optimality of the weights

The PRCC is thus a measure for the aggregate imbalance between the component performance and risk contributions. In this section, we show that it is also a measure for the sensitivity of the portfolio performance and risk with respect to small changes in the portfolio weights. In fact, when the portfolio weights are determined on the basis of performance and risk, they should be seen as estimates, due to the estimation uncertainty in the portfolio performance and risk. There is thus a mismatch between the actual weight vector \mathbf{w} and the weight vector the investor would choose if there were no estimation error. Let us denote the latter by \mathbf{w}^b . It is desirable that the estimation error in the portfolio weights has only limited influence on the portfolio performance and risk, and thus that $\mathcal{P}_p(\mathbf{w}) \approx \mathcal{P}_p(\mathbf{w}^b)$ and $\mathcal{R}_p(\mathbf{w}) \approx \mathcal{R}_p(\mathbf{w}^b)$. We formalize below the impact of incremental weight perturbations and show that the PRCC can be directly interpreted as a measure of robustness of the portfolio performance and risk to those weights' perturbations.

We use the additive perturbation model to quantify the effect of small errors on the portfolio performance. It consists of considering the change in the portfolio performance and risk when the portfolio weight vector \mathbf{w} changes to $\tilde{\mathbf{w}} \equiv \mathbf{w} + \varepsilon \mathbf{e}_i$, with ε an infinitesimal small positive number and \mathbf{e}_i the i th basis vector of dimension $N \times 1$.

It is desirable that this small perturbation has little or no impact on the evaluation of the performance/risk contribution of the different assets in the portfolio. In other words, that:

$$\frac{\partial \mathcal{P}_p(\mathbf{w})}{\partial w_i} - \tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w}) \frac{\partial \mathcal{R}_p(\mathbf{w})}{\partial w_i} \approx 0.$$

Such a property would indicate robustness of the portfolio performance and risk to estimation error

in the weights (*i.e.*, $\mathcal{P}_p(\mathbf{w}) \approx \mathcal{P}_p(\mathbf{w}^b)$ and $\mathcal{R}_p(\mathbf{w}) \approx \mathcal{R}_p(\mathbf{w}^b)$).

By the definitions of the component performance and risk contributions in (2) and (3), this desire for robustness of the performance to small perturbations in the portfolio weights is equivalent to the condition that the PRCC is close to zero.

As such, we thus obtain a multiple-criteria portfolio optimization problem: Maximize performance, minimize risk and minimize PRCC.³ In the next section, we recommend to simplify the problem by considering only preferences for the value of the portfolio's relative performance (as a summary for the balance between maximum performance and minimum risk) and the value of the PRCC. In the special case of maximizing the relative performance, the PRCC is zero and there is no trade-off between the two objectives. In fact, as we show in Appendix B, for the maximum relative performance portfolio, we have that the performance and risk contributions are optimally aligned in the sense of an equality between the performance contribution and the risk contribution, scaled by the portfolio's relative performance ratio:

$$C_i^{\mathcal{P}}(\mathbf{w}^*) = \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) C_i^{\mathcal{R}}(\mathbf{w}^*), \quad (7)$$

for all $i = 1, \dots, N$ and where $\mathbf{w}^* \equiv \operatorname{argmax}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w})$, with $\mathcal{C}_{\text{FI}} \equiv \{\mathbf{w} \in \mathbb{R}^N \mid \mathbf{w}'\mathbf{1} = 1\}$ the set of portfolio weights satisfying the full-investment constraint. Large values of the PRCC thus indicate active bets in terms of deviating performance/risk contributions from those of the maximum relative performance portfolio.

³There exist several alternative approaches for reducing the impact of estimation error in the optimized portfolio weights. The first one is to use shrinkage methods or resampling approaches in the estimation (see, *e.g.*, Ledoit and Wolf, 2008; Michaud and Michaud, 2008). A second solution is to impose bound constraints on the portfolio weights (see, *e.g.*, Jagannathan and Ma, 2003).

2.3. The PRCC diagram for visualizing intra-portfolio alignment of performance and risk contributions

As argued above, it is desirable that the portfolio relative performance is not highly sensitive to small weight perturbations. We find it useful to visualize this sensitivity in a scatter plot of the component risk contributions against the component performance contributions, together with the line through the origin with slope equal to the portfolio's relative performance. The portfolio components that are on this line correspond to positions for which the alignment at the position level between risk and performance is identical to the alignment at the aggregate portfolio level.

As an illustration, consider a portfolio of four assets. Their performance contributions and risk contributions are given by 1.4, 1.6, 1.7, 2.1 and 1, 2, 2.5, 3, respectively. Figure 1 shows the corresponding PRCC diagram. In this example, the portfolio aggregate performance and risk values are 6.8 and 8.5, respectively. The portfolio relative performance is therefore 0.8 (unit of performance per unit of risk). The portfolio components that are on this line correspond to positions for which the alignment at the position level between performance and risk is identical to the alignment at the aggregate portfolio level. In our example, this is the case for asset 2. When the performance/risk contribution couple is above the line, the position contributes more to portfolio performance per unit of risk than the total portfolio (*e.g.*, asset 1), and vice versa when the couple is below the line (*e.g.*, assets 3 and 4). At the aggregate portfolio level, the sum of all CPRC is by construction exactly zero as stated by (4). The PRCC value is 0.14.

[Insert Figure 1 about here]

3. Choice of risk measure and the PRCC

In this section we first focus on the volatility as the risk measure and derive alternative representations of the PRCC for specific risk-based portfolios. We then discuss the implementation of the PRCC with downside risk measures.

3.1. The volatility-based PRCC

The Sharpe ratio is certainly the most often used performance measure in portfolio analysis. As an *ex-post* measure of performance, it reflects the excess return compensation received per unit of volatility risk taken. To be an *ex-ante* target, we need the forecast of portfolio return and volatility. More formally, we consider a portfolio invested in N assets with vector of weights $\mathbf{w} \equiv (w_1, \dots, w_N)'$. We denote $\boldsymbol{\mu} \equiv (\mu_1, \dots, \mu_N)'$ as the vector of expected (arithmetic) returns and $\tilde{\boldsymbol{\mu}} \equiv (\tilde{\mu}_1, \dots, \tilde{\mu}_N)'$ as the vector of expected excess (arithmetic) returns over the risk-free rate. We further define the $N \times N$ covariance matrix of (arithmetic) returns by $\boldsymbol{\Sigma}$. Then the portfolio's expected excess return can be written as $\tilde{\mu}_p(\mathbf{w}) \equiv \mathbf{w}'\tilde{\boldsymbol{\mu}}$ and the portfolio volatility is given by $\sigma_p(\mathbf{w}) \equiv \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$. Both measures are first-degree homogeneous. For the Sharpe ratio, the performance contribution of asset i is given by:

$$C_i^\mu(\mathbf{w}) \equiv w_i \tilde{\mu}_i,$$

and the component volatility contribution of asset i is given by:

$$C_i^\sigma(\mathbf{w}) \equiv w_i \partial_i \sigma_p(\mathbf{w}) = w_i \frac{[\boldsymbol{\Sigma}\mathbf{w}]_i}{\sigma_p(\mathbf{w})}.$$

The PRCC for the portfolio with a Sharpe ratio target $\tau_p^{\mu,\sigma}(\mathbf{w})$ is given by:

$$\text{PRCC}(\mathbf{w}) \equiv \frac{1}{N} \sum_{i=1}^N [C_i^\mu(\mathbf{w}) - \tau_p^{\mu,\sigma}(\mathbf{w}) C_i^\sigma(\mathbf{w})]^2. \quad (8)$$

The general expression of the PRCC in (8) evaluates the mismatch between the performance and risk contributions of a fully invested portfolio with weights \mathbf{w} . In Appendix C, we derive specific formulas of the volatility-based PRCC for the minimum downside risk portfolio⁴, the minimum

⁴The portfolio minimizes a downside risk measure belonging to the family of risk measures of the form $\mathbf{w}'\boldsymbol{\mu} -$

variance portfolio, the inverse volatility portfolio, the equally-weighted portfolio, the equal-risk-contribution portfolio and the maximum diversification portfolio. The resulting expressions for the PRCC are presented in Table 1.

[Insert Table 1 about here]

The analysis shows that, for the minimum downside risk portfolio, the value of the PRCC is independent of the cross-sectional variation in the expected returns. It is only the total portfolio risk that determines the value of the portfolio's PRCC. In contrast, for the minimum variance and equal-risk-contribution portfolios, the percentage risk contributions have no influence on the PRCC, wthe value of which is a function of the percentage return contributions. Finally, we see in Table 1, that for the maximum diversification portfolio, the PRCC is a function of the variability of the spread between the assets' individual Sharpe ratios and the ratio between the weighted average return and the weighted average volatility, with weights corresponding to the portfolio weights.

3.2. Downside risk-based PRCC

The general definition of the PRCC requires first-degree homogeneous functions of performance and risk. For the former, we use the portfolio excess return throughout the paper. For the latter, we investigate here the use of a downside risk measure. We hereby follow the convention of referring to downside risk as a positive number and using low-probability terminology (see, *e.g.*, Daniélsson, 2011, Subsection 4.3.1). We focus the analysis on downside risk measures that can be written as a linear combination of the portfolio expected return and volatility:

$$\mathcal{R}_p(\mathbf{w}) \equiv -\mu_p(\mathbf{w}) + c_p(\mathbf{w})\sigma_p(\mathbf{w}),$$

where $c_p(\mathbf{w})$ depends on the distribution of portfolio returns. Table 2 presents common choices of $c_p(\mathbf{w})$ leading to an estimator of the portfolio VaR and ES, obtained when assuming either a

$c_Z\sqrt{\mathbf{w}'\Sigma\mathbf{w}}$, such as the VaR and ES of the portfolio under the assumption of an elliptically symmetric distribution.

Gaussian distribution, a Student- t distribution, the Cornish-Fisher approximation, or the historical estimation methodology (see, *e.g.*, Praetz, 1972; Boudt et al., 2008; Martin and Arora, 2015). Throughout the paper, we use VaR and ES computed at the 5% loss probability.

[Insert Table 2 about here]

4. Optimizing the PRCC of a risk-based portfolio

By definition, the weights of a risk-based portfolio do not depend on an estimate of expected returns. They thus have the advantage that estimation error in the (noisy) expected return estimates has no influence on the portfolio weights. Their weakness lies in the model risk implied, since, as shown by Ardia and Boudt (2015) among others, it is only under specific conditions on the expected returns that the risk-based portfolio is an efficient investment decision. Moreover, when the risk-based portfolio weights are based on estimates of the risk parameters (such as the covariance or the Value-at-Risk), they are still sensitive to estimation error. For this reason, we propose here to reach a compromise between the advantages of risk-based portfolios, and their disadvantages, by using the PRCC criterion to modify the risk-based portfolio weights. The modification aims at improving the weights by tilting them in the direction for which the performance and risk contributions are better aligned. By doing so, the weights violate less the first-order condition of the maximum Sharpe ratio portfolio (implying a zero value for the PRCC) and are more robust to estimation error, as explained in Section 2.2. To preserve the interpretation of the risk-based portfolio weights and limit the impact of estimation error in the expected returns, we restrict the weight modifications in two ways. First, we require that the weights' modification does not alter the relative performance of the portfolio compared with the risk-based benchmark. Second, we impose an upper-bound constraint on the mean squared value of the weight differences induced by the optimization, and refer to this as the tracking error constraint.

More formally, our optimization framework considers an investor who has a risk-based reference portfolio w^* with relative performance equal to τ_p^* . The investor seeks to optimize the

portfolio weights by aligning his performance and risk contributions (*i.e.*, minimizing his portfolio's PRCC), under the constraint that the portfolio relative performance ratio equals τ_p^* and the optimized portfolio weights are sufficiently close to the weights of the reference portfolio.

The first constraint thus determines the balance between performance and risk at the aggregate portfolio level by setting a target value on the relative performance ratio. In our context of aligning the performance and risk contributions, imposing a target value for the relative performance ratio is natural, since it fixes the parameter τ_p in the PRCC criterion and therefore facilitates the identification of portfolios for which the performance and risk contributions are matched. The target value for τ_p is of course crucial, as it determines the balance between performance and risk at the aggregate portfolio level. The higher this target level is, the more risky the optimized portfolio tends to be. An important caveat is that, in the presence of estimation error, it can be expected that the higher the target level is, the more sensitive the optimized portfolio is to estimation error.

The second constraint can be interpreted as a tracking error constraint on the active weights. As in Brandt et al. (2009), we write this tracking error constraint on the portfolio weights as:

$$\frac{1}{N} \sum_{i=1}^N (w_i - w_i^*)^2 \leq \zeta^2,$$

where we set ζ at 10% for the base case in our empirical application. Under this constraint, the PRCC optimization can thus be seen as shrinking the predetermined risk-based portfolio weights towards a state in which the performance and risk contribution portfolios are aligned. The constraint limits the impact of estimation error on the optimized portfolio weights, as in Bera and Park (2008), and ensures that the optimized weights can be interpreted as enhancements to the risk-based portfolio weights.

In addition to the target value constraint for the portfolio relative performance and the tracking error constraint, we require the portfolio to be fully invested and we do not allow for short positions. This leads to the following optimization problem, taking the risk-based portfolio weights

\mathbf{w}^* as input, and transforming them into PRCC modified risk-based portfolio weights:

$$\begin{aligned}
& \underset{\mathbf{w}}{\text{minimize}} && \text{PRCC}(\mathbf{w}) \\
& \text{subject to} && \tau_p(\mathbf{w}) = \tau_p^* \\
& && \sqrt{\frac{1}{N} \sum_{i=1}^N (w_i - w_i^*)^2} \leq \zeta \\
& && \mathbf{w}'\mathbf{1} = 1 \\
& && w_i \geq 0 \quad \forall i \\
& && w_i < 1 \quad \forall i.
\end{aligned} \tag{9}$$

The strict inequality constraint that $w_i < 1$ ensures that the portfolios are invested in at least two assets (see Footnote 2 on page 7).

From a computational viewpoint, the derivation of the PRCC modified risk-based portfolio weights is a nonlinear optimization problem. When the PRCC has smooth first and second order derivatives (with respect to \mathbf{w}), this problem can be easily solved using sequential quadratic programming. This is the case when the performance measure is the portfolio mean and the portfolio risk measure is the portfolio volatility or the VaR (or ES) computed under the assumption of the Gaussian or Student- t distribution, or when using the Cornish-Fisher approximation (*i.e.*, modified VaR and modified ES).⁵

5. Illustration in asset allocation

Risk-based portfolios are increasingly used in the construction of equity portfolios and in asset allocation. For sake of clarity in our presentation, we choose to illustrate the use of the PRCC in asset allocation, because of the typically lower dimension of a realistic asset allocation portfolio compared with a realistic equity optimization problem. Our goal is to determine the weights of the

⁵In the application, we optimize our portfolios using the R package **donlp2** (Tamura, 2007).

portfolio invested in six asset classes: (i) developed markets equity, (ii) emerging markets equity, (iii) US government bond, (iv) US investment grade corporate bond, (v) real estate and (vi) gold.

We start the illustration by introducing the monthly return data used for the period 1988-2015. We then use the PRCC to analyze and modify six risk-based portfolios: (1) the minimum downside risk portfolio, (2) the minimum variance portfolio, (3) the inverse volatility portfolio, (4) the equally-weighted portfolio, (5) the equal-risk-contribution portfolio, and (6) the maximum diversification portfolio. The main analysis uses volatility as a risk measure and the Sharpe ratio as relative performance measure. To illustrate the use of a downside risk measure, we also consider the modified VaR as risk measure, together with the corresponding modified Sharpe ratio as relative performance measure. We first present our in-sample results and conclude with an extensive out-of-sample performance evaluation using rolling estimation windows of three years and monthly portfolio rebalancing. Throughout the analysis, we obtain the estimates for the expected (excess) returns and covariance matrix using the standard sample mean and covariance of the monthly arithmetic (excess) returns of the six assets.⁶

5.1. Data

The sample ranges from January 1988 to August 2015. We use the end-of-month values on the total return index of the MSCI World index, the MSCI Emerging Markets index, Bloomberg US Government bond (1-10 year) index, BofA Merrill Lynch US Corp Master Total return index, All REITS Total index and Gold Fixing price 3 p.m (London time) in the London Bullion Market. All returns computed are arithmetic returns, based on the USD value of the indices. We take the US one-month Treasury bill rate from the database of Kenneth French as the risk-free asset.⁷

⁶The use of rolling estimation windows reflects industry practice when the frequency of rebalancing is monthly. Alternatively, more complex estimators could be considered that use higher frequency data (see e.g. Boudt and Zhang (2015)) or by considering a parametric approach to modelling the time-variation in the return series (see e.g. Boudt et al. (2012)).

⁷The MSCI World index tracks the performance of large and mid-cap equities over 23 developed market countries: <https://www.msci.com/market-cap-weighted-indexes>. The data of All REITS Total index is retrieved from: <https://www.reit.com/investing/index-data/>

The summary statistics on the buy-and-hold investments in each of six assets are reported in Table 3. Among the six assets considered, the MSCI Emerging Markets index has the highest annualized excess return (9.77%), followed by the NAREIT index (7.55%). On the risk side, the Bloomberg US Government Bond index has the lowest level of annualized volatility (3.16%) and drawdown (3.5%). Over the period, the two bond indices have the most attractive Sharpe ratio (0.23). The investment in the gold index is the least attractive in terms of average return and Sharpe ratio performance. In some cases, it may be valuable to include an investment in gold to the portfolio, because, as can be seen in Panel B of Table 3, it is a great diversifier in the portfolio. Its correlation with the five other asset classes is below 0.2. The highest correlation is observed between the returns of the two equity indices (0.73). Finally, note that the US Government bond had negative correlations with the MSCI World index (-0.09) and the MSCI Emerging market index (-0.15) over our period, which is marked by the financial crisis and possible flights to safety.

[Insert Table 3 about here]

5.2. *In-sample performance and PRCC of risk-based portfolios*

Given the large heterogeneity in performance of the various asset classes, it is now relevant to study how the choice of risk-based portfolio allocation affects portfolio performance. The results are reported in Panel A of Table 4. In terms of in-sample annualized returns, we see that the equally-weighted portfolio has the highest return (around 5%), while the minimum variance portfolio offers only an average return of 2.66%. The equal-risk-contribution portfolio is the second best in terms of annualized returns (3.58%). Its volatility is 4.55%, which is in between the 3.01% volatility of the minimum variance portfolio and the 8.92% volatility of the equally-weighted portfolio. The Sharpe ratio of the maximum Sharpe ratio portfolio is twice the Sharpe

monthly-index-values-returns. For the US Corp Master Total return index and gold spot, the data is collected from the Federal Reserve Bank of St. Louis, while the risk-free rate data used is the one from K. French data website, available at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

ratio of the equally-weighted portfolio. In our sample, maximizing the Sharpe ratio and minimizing the downside risk or variance leads to similar portfolios with a high allocation to bonds. The maximum Sharpe ratio portfolio allocates to both US government bonds (85%) and investment grade corporate bonds (5%), while the minimum downside risk portfolio and minimum variance portfolio only invest in the government bonds (93%).

[Insert Table 4 about here]

The results on portfolio performance are as expected. The main novelty in Table 4 is the column with PRCC[†] values, for which high values indicate concentrated bets in terms of volatility and expected return contributions that are not aligned with the portfolio's Sharpe ratio. We see that the minimum variance and minimum downside risk portfolios are close to optimal in terms of a low value of the PRCC, while the inverse volatility, equally-weighted, equal-risk-contribution and maximum diversification portfolios have a monthly PRCC value that is higher than 0.00075. In annualized terms, this corresponds to a PRCC value of 0.11, as obtained by multiplying the monthly PRCC value with 144 (*i.e.*, 12^2). Table 5 shows the annualized performance and risk contributions that lead to the annualized value of the PRCC. Interestingly, we see that for the four mentioned portfolios with a large PRCC value, this is caused by the investments in the equity, NAREIT and Gold asset classes, which cause too much volatility compared with their return contribution, while the position in bonds contributes relatively more to return than it does to risk. Because of the nonlinear dependence of the PRCC on portfolio weights, it is not possible to predict how the PRCC modification will change the weights in the individual position. We analyze this empirical question next.

[Insert Table 5 about here]

5.3. *In-sample gains from optimizing the PRCC*

Let us now investigate how the PRCC modification to risk-based weights changes the portfolio performance and weights. Taking volatility as our risk measure and setting $\zeta = 10\%$, the in-sample

performance is shown in Panel B of Table 4, while the weights of the PRCC optimized portfolios are shown in the right part of Table 5. Remember also that an important constraint in the PRCC modification is that the portfolio needs to have the same Sharpe ratio, as can be seen in Table 4. It follows that the PRCC modification comes either at the price of a higher volatility (which is the case for the minimum downside risk, minimum variance, the equally-weighted portfolio) or a lower return (which is the case for the equally-weighted and maximum diversification portfolio). Another constraint is the 10% upper bound on the tracking error in terms of weights compared to the benchmark portfolio. We see in the column ‘TE’ that this constraint is binding for the inverse volatility weighted, the equal-risk-contribution and the max diversification portfolios. Finally, it is of interest to see that the direction of the weight changes due to the PRCC modification depends on the risk-based portfolio considered. For the equally-weighted portfolio, we see, *e.g.*, that a better equilibrium between the performance and risk contributions is obtained by overweighting equities and real estate, while for the maximum diversification portfolio, the weight to bonds is substantially increased. In case of the equal-risk-contribution portfolio, we see that the PRCC modified portfolio invests approximately the same weights in bonds, but instead of concentrating 50% of the weight in the government bond and only 20% in the corporate bonds, the PRCC modified equal-risk-contribution portfolio invests around 35% in both of them.

[Insert Figure 2 about here]

We further visualize these changes in performance and risk contribution in Figure 2 for the equally-weighted and equal-risk-contribution portfolio. Note that the slope of the line in each plot corresponds to the portfolio’s Sharpe ratio. The steeper the line, the higher the value of the Sharpe ratio. Henceforth, we call this the Sharpe ratio allocation line, since each point corresponds to a performance contribution that equals the portfolio’s Sharpe ratio multiplied with its risk contribution.

Consider first the upper plot, showing that for the equally-weighted portfolio, the annualized

Sharpe ratio equals 0.56. In spite of the equal weight allocation, the performance and risk contributions are heterogeneous across the stocks, with occasional large deviations from the line corresponding to a performance contribution that equals the portfolio's Sharpe ratio, multiplied with the volatility contribution. Interestingly, the PRCC modification tends to lead to more dispersed risk and performance contributions, but they are better aligned and on average closer to the Sharpe ratio allocation line.

In the bottom plot, we see that, in case of the equal-risk-contribution portfolio, the risk contributions are of course identical for all assets, leading to a vertical line of performance contributions, and a high value for the PRCC. The PRCC modification of the equal-risk-contribution portfolio leads to portfolio weights with heterogeneous risk contributions across the assets, and performance contributions that are more aligned to the volatility contributions, since the value of the annualized PRCC drops from 0.12 to 0.06.

Let us now consider two more robustness analyses in Panels C and D of Table 4. The first one is with respect to the upper bound on the tracking error of the PRCC optimized portfolios compared with their benchmark. In the main analysis, the bound ζ is set to 10%, which is binding for the inverse volatility weighted, equally-weighted, equal-risk-contribution and maximum diversification portfolios. In Panel C we restrict the PRCC optimization even further by setting $\zeta = 5\%$. Obviously, the resulting PRCC is higher than in the case of $\zeta = 10\%$. However, even allowing for relatively minor changes in the weights, we see that the PRCC can be substantially reduced.

Finally, in Panel D, we consider the case of an investor who used the Cornish-Fisher modified VaR (mVaR) as the measure of risk in quantifying the portfolio's PRCC and its relative performance in terms of the modified Sharpe ratio. Compared with the benchmarks in Panel A, we see that, except for the inverse volatility portfolio, all other PRCC modified risk-based portfolios have higher average returns (2.8%-5.8% versus 2.6%-5.0%) at the price of a higher modified VaR (1.01%-4.62% versus 0.96%-3.99%). For these portfolios, their volatilities are also higher than the benchmarks (3.12%-10.41% versus 3.01%-8.92%). The 10% tracking error constraint is binding

for the inverse volatility, equal-risk-contribution and maximum diversification portfolios. Comparing Panel B with Panel D, we find that the effect of deciding to optimize the PRCC has a larger effect in our sample than the choice of using modified VaR instead of volatility in the definition of the PRCC and relative performance measure.

5.4. Out-of-sample gains from optimizing the PRCC

To assess the out-of-sample performance from optimizing the PRCC, we implement an investment strategy that rebalances the portfolios at the end of the month. At each rebalancing date, all parameters needed for the calculation of the PRCC and the portfolio optimization are estimated using the 36 most recently observed monthly returns (*i.e.*, on a rolling-window basis). The relative performance targets are set equal to those of the risk-based portfolio rules. In terms of performance measures, we report, for all strategies: (i) the cumulative value, (ii) the annualized geometric return, (iii) the annualized excess return, (iv) the annualized volatility, (v) the Sharpe ratio, (vi) the portfolio skewness, (vii) the portfolio kurtosis, (viii) the maximum drawdown, (ix) the 5% modified VaR, (x) the corresponding modified Sharpe ratio, and (xi) the average of the tracking error of the PRCC modified portfolios compared with their risk-based benchmark portfolio. The out-of-sample period ranges from January 1991 to August 2015 for a total of 296 monthly observations.

Results are presented in Table 6. Consider first the out-of-sample performance of the benchmark risk-based portfolios in Panel A, and compare them with the maximum (modified) Sharpe ratio portfolios in Panel E. We find that the risk-based portfolios are successful in achieving the proposed investment style out-of-sample. As predicted, the minimum variance portfolio and minimum downside risk portfolio have the lowest level of volatility and drawdown (3% and 5%, respectively), and that the volatility of the equal-risk-contribution portfolio (5.85%) is in between the volatility of the minimum variance portfolio (2.95%) and the equally-weighted portfolio (9.12%).

We further observe in Table 6 that, for our sample, the highest Sharpe ratio is not achieved by the maximum Sharpe ratio portfolios in Panel E, but by the low risk portfolios (minimum down-

side risk and minimum variance portfolios) in Panel A. This can be explained by the presence of estimation error in the expected returns, leading to a poor out-of-sample performance of the maximum relative performance portfolios. The PRCC modified risk-based portfolios are partially safeguarded against the estimation risk because of the balancing objective between performance and risk contributions, and the tracking error constraint on the portfolio weights. In terms of absolute performance, the equally-weighted portfolio has the highest end-value (\$7.01 for \$1 invested in 1991). It also has the highest drawdown (31.64%) of all portfolios considered.

[Insert Table 6 about here]

Panel B of Table 6 shows that the PRCC modification leads to a substantial improvement in the performance of the equally-weighted portfolio. The annualized geometric return increases from 8.22% to 8.76%, while its annualized volatility and max drawdown decrease from 9.12% and 31.64% to 8.17% and 27.95%, respectively. This performance improvement effect is consistent with the expectation in Kritzman et al. (2010) that optimization must be able to improve performance of a naively diversified portfolio, like the equally-weighted portfolio. For the other risk-based portfolios, the effect of the PRCC modification is to increase absolute performance at the cost of an increase in risk. This trade-off effect is in line with the constraint on the equality of estimated Sharpe ratio between the traditional risk-based portfolio and the PRCC modified portfolio. Note also in the last column ("TE") that the PRCC modification leads to the smallest weight changes for the minimum downside risk and minimum variance portfolios, while for the equally-weighted portfolio, the average tracking error is 9.98, indicating that for almost all rebalancing dates, the upper 10% constraint is binding.

In Panel C of Table 6 we investigate the effect of restricting the tracking error constraint further by setting ζ to 5%. Such a stricter constraint is desirable when the portfolio managers sets a higher priority to similarity of the PRCC modified weights to the risk-based weights. We see that the resulting performance is in between the benchmark risk-based portfolios and the PRCC modified

risk based portfolio with $\zeta = 10\%$. The main conclusions still hold, namely that the PRCC modification leads to an improvement of both performance and risk of the equally-weighted portfolio, while for the other portfolios, there is an improved absolute performance at the price of a higher risk. The PRCC modification either leads to a similar or improved relative performance.

All results discussed above are when the PRCC is implemented using volatility as the risk measure and the Sharpe ratio as the relative performance measure. Panel D of Table 6 reports the out-of-sample performance when the 5% modified Value-at-Risk is used as risk measure, together with the corresponding modified Sharpe ratio. When considering the modified VaR, we now see that the PRCC modification increases performance and decreases the downside risk for the inverse volatility weighted, equally-weighted and maximum diversification portfolio. For the other portfolios, the absolute performance improves at the price of slightly higher modified Value-at-Risk, but the relative performance is either similar or better.

The bottom line results of the out-of-sample study can thus be summarized as follows. Firstly, the PRCC modification improves the performance of the equally-weighted portfolio on all performance dimensions considered. For the other risk-based portfolios, the PRCC modification improves the absolute performance, but typically also increases the risk. These results are robust to tightening the tracking error constraint from $\zeta = 10\%$ to $\zeta = 5\%$, and to the use of modified Value-at-Risk rather than volatility as the risk measure.

6. Conclusion

Risk-based portfolios have the computational and practical advantages of not requiring a return forecast. We show that this may lead to imbalances in terms of a disparity between the performance contribution per unit of risk contribution for the various portfolio positions. To measure this imbalance, we propose the Performance/Risk Contribution Concentration (PRCC) measure. It has the additional interpretation of measuring the robustness of the portfolio performance and risk with

respect to small weight perturbations. We also show how to improve the balance between the performance and risk contributions of a reference portfolio by minimizing the portfolio PRCC under the constraint of achieving the same relative performance, and that the portfolio weights must be close enough to the benchmark weights. The proposed PRCC modified risk-based portfolio has the potential to strike a balance between investors who believe in the construction of optimized portfolios using return forecasts (see, *e.g.*, Kritzman et al., 2010), and investors who emphasize the difficulty in estimating expected returns and recommend to use portfolio allocations that do not require expected return estimates (see, *e.g.*, DeMiguel et al., 2009).

We analyze the usefulness of the PRCC for the asset allocation decision to invest among equities, bonds, real estate, and gold. We find that, of all portfolios considered, the inverse volatility weighted, equally-weighted, equal-risk-contribution, and maximum diversification portfolios have the highest value of PRCC. Optimizing the PRCC of risk-based portfolios is especially beneficial when considering the equally-weighted portfolio: it increases the performance and reduces the risk. For the other risk-based portfolios, we find that the PRCC modification tends to yield similar or improved values for the relative performance, by increasing total performance at the price of a higher risk.

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Table 1: Simplified representation for the volatility-based PRCC measure of risk-based portfolios

This table presents expressions of the PRCC for six widely used risk-based portfolios. Details of the formulas can be found in Appendix C. We use $\boldsymbol{\sigma} \equiv (\sigma_1, \dots, \sigma_N)'$, $\boldsymbol{\xi} \equiv (1/\sigma_1, \dots, 1/\sigma_N)'$, $\boldsymbol{\iota}$ is a $N \times 1$ vector of ones, $\boldsymbol{\Sigma}$ and \mathbf{R} are the $N \times N$ covariance and correlation matrices, respectively.

Portfolio rule	PRCC(\mathbf{w}^*)
<i>Minimum downside risk portfolio</i>	
$\mathbf{w}^* \equiv \operatorname{argmin}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \left\{ -\mathbf{w}'\boldsymbol{\mu} + c_Z \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}} \right\}$	$\frac{1}{N} \cdot \frac{[\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z]^2}{\sigma_p^2(\mathbf{w}^*)} \cdot \sum_{i=1}^N \left\{ w_i^* [\sigma_p^2(\mathbf{w}^*) - [\boldsymbol{\Sigma}\mathbf{w}^*]_i] \right\}^2$
<i>Minimum variance portfolio</i>	
$\mathbf{w}^* \equiv \operatorname{argmin}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \left\{ \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \right\}$	$\frac{1}{N} \cdot \sum_{i=1}^N \left\{ w_i^* [\tilde{\mu}_i - \tilde{\mu}_p(\mathbf{w}^*)] \right\}^2$
<i>Inverse volatility portfolio</i>	
$\mathbf{w}^* \equiv \boldsymbol{\xi}/\boldsymbol{\xi}'\boldsymbol{\iota}$	$\frac{1}{N} \cdot \frac{1}{(\boldsymbol{\xi}'\boldsymbol{\iota})^2} \cdot \sum_{i=1}^N \left[\frac{\tilde{\mu}_i}{\sigma_i} - \tau_p^{\mu, \sigma}(\mathbf{w}^*) \frac{[\mathbf{R}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}} \right]^2$
<i>Equally-weighted portfolio</i>	
$\mathbf{w}^* \equiv \boldsymbol{\iota}/N$	$\frac{1}{N^3} \cdot \sum_{i=1}^N \left[\tilde{\mu}_i - \tau_p^{\mu, \sigma}(\mathbf{w}^*) \frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}} \right]^2$
<i>Equal-risk-contribution portfolio</i>	
$\mathbf{w}^* \equiv \operatorname{argmin}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \left\{ \sum_{i=1}^N \sum_{j=1}^N [C_i^\sigma(\mathbf{w}) - C_j^\sigma(\mathbf{w})]^2 \right\}$	$\frac{1}{N} \cdot \sum_{i=1}^N \left[w_i^* \tilde{\mu}_i - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N} \right]^2$
<i>Maximum diversification portfolio</i>	
$\mathbf{w}^* \equiv \operatorname{argmax}_{\mathbf{w} \in \mathcal{C}_{\text{FI}}} \left\{ \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \right\}$	$\frac{1}{N} \cdot \sum_{i=1}^N \left[w_i^* \sigma_i \left(\frac{\tilde{\mu}_i}{\sigma_i} - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{\boldsymbol{\sigma}'\mathbf{w}^*} \right) \right]^2$

Table 2: Expressions for the volatility multiplier

The table presents expressions for the volatility multiplier $c_p(\mathbf{w})$ and its partial derivative $\partial_i c_p(\mathbf{w})$ for various downside risk measures of the form $\mathcal{R}_p(\mathbf{w}) \equiv -\mu_p(\mathbf{w}) + c_p(\mathbf{w})\sigma_p(\mathbf{w})$. z_α is the α -quantile and $\phi(\cdot)$ is the density function of the standard Gaussian distribution. $T_{\alpha,\nu}^{-1}$ is the α -quantile and $t_\nu(\cdot)$ is the density of the Student- t distribution with ν degrees freedom. $g_\alpha(\mathbf{w})$ is the α -quantile obtained with the Cornish-Fisher expansion. The portfolio skewness and kurtosis are denoted by $s_p(\mathbf{w})$ and $k_p(\mathbf{w})$, respectively. Expressions for the partial derivatives of the skewness and kurtosis are presented in Appendix D. The historical VaR estimation is detailed in Appendix E.

Downside risk measure	$c_p(\mathbf{w})$	$\partial_i c_p(\mathbf{w})$
<i>Panel A: Value-at-Risk (VaR)</i>		
Gaussian VaR (GVaR)	$-z_\alpha$	0
Student- t VaR (tVaR)	$-T_{\alpha,\nu}^{-1} \sqrt{\frac{\nu-2}{\nu}}$	0
Modified VaR (mVaR)	$-g_\alpha(\mathbf{w})$	$-\partial_i g_\alpha(\mathbf{w})$
Historical VaR (HVaR)	$\frac{\text{HVaR}_\alpha + \mu_p(\mathbf{w})}{\sigma_p(\mathbf{w})}$	$\frac{1}{\sigma_p(\mathbf{w})} \left[\frac{c_i^{\text{HVaR}_\alpha}(\mathbf{w})}{w_i} + \mu_i - c_p(\mathbf{w}) \frac{[\Sigma \mathbf{w}]_i}{\sigma_p(\mathbf{w})} \right]$
<i>Panel B: Expected Shortfall (ES)</i>		
Gaussian ES (GES)	$\frac{1}{\alpha} \phi(z_\alpha)$	0
Student- t ES (tES)	$\frac{1}{\alpha} \sqrt{\frac{\nu}{\nu-2}} t_\nu \left(\sqrt{\frac{\nu-2}{\nu}} T_{\alpha,\nu}^{-1} \right)$	0
Modified ES (mES)	$\frac{1}{\alpha} \phi(g_\alpha(\mathbf{w})) \left[1 + \frac{1}{6} g_\alpha^3(\mathbf{w}) s_p(\mathbf{w}) + \frac{1}{72} (g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3) s_p^2(\mathbf{w}) + \frac{1}{24} (g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1) k_p(\mathbf{w}) \right]$	See Appendix D
Historical ES (HES)	$\frac{\text{HES}_\alpha + \mu_p(\mathbf{w})}{\sigma_p(\mathbf{w})}$	$\frac{1}{\sigma_p(\mathbf{w})} \left(\frac{c_i^{\text{HES}_\alpha}(\mathbf{w})}{w_i} + \mu_i - c_p(\mathbf{w}) \frac{[\Sigma \mathbf{w}]_i}{\sigma_p(\mathbf{w})} \right)$

Table 3: Descriptive statistics of asset returns

This table presents summary statistics of the monthly returns for the six assets in our universe. In Panel A, we report the cumulative terminal value of a \$1 investment (\$), the annualized geometric returns (GR, in percent), the annualized excess returns (Mean, in percent), annualized standard deviation (Sd, in percent), Sharpe ratio (SR), skewness (Sk), kurtosis (Ku), the maximum drawdown (MDD, in percent), the 5% modified Value-at-Risk (mVaR, in percent), and the corresponding modified Sharpe ratio (mSR). Panel B reports the correlation between the monthly asset returns. The signs ***, ** and * indicate whether the Pearson correlation coefficient is significantly different from zero at the 1%, 5% and 10% levels, respectively. The six asset classes considered are the MSCI World index-developed countries (Eq-DE), the MSCI Emerging markets index (Eq-EM), the US Government bond index (Bo-GO), the US corporate bond master index (Bo-CO), NAREIT, and the Gold spot index (Gold). The sample period ranges from January 1988 to August 2015 for a total of 332 monthly observations.

<i>Panel A: Buy-and-hold strategy</i>										
Asset	\$	GR	Mean	Sd	SR	Sk	Ku	MDD	mVaR	mSR
Eq-DE	7.53	7.57	5.20	14.94	0.10	-0.61	1.37	53.65	6.97	0.06
Eq-EM	16.78	10.73	9.77	23.30	0.12	-0.59	1.64	61.44	10.83	0.08
Bo-Go	4.75	5.80	2.45	3.16	0.22	-0.01	0.23	3.46	1.02	0.20
Bo-Co	7.21	7.40	4.05	5.19	0.23	-0.80	4.30	16.07	2.05	0.16
NAREIT	12.71	9.62	7.55	17.48	0.12	-0.84	8.00	67.89	7.71	0.08
Gold	2.34	3.13	1.06	15.69	0.02	0.13	1.22	47.37	6.79	0.01

<i>Panel B: Correlation between assets</i>					
Asset	Eq-DE	Eq-EM	Bo-Go	Bo-Co	NAREIT
Eq-EM	0.73***				
Bo-Go	-0.09	-0.15***			
Bo-Co	0.29***	0.22***	0.68***		
NAREIT	0.54***	0.45***	0.01	0.36***	
Gold	0.05	0.16***	0.10*	0.15***	0.06

Table 4: In-sample performance and tracking error results

This table presents the in-sample performance and tracking error results for the benchmark risk-based portfolios, the PRCC modified risk-based portfolios and the maximum Sharpe ratio portfolio. Panel A shows portfolio performance of the risk-based portfolios. Panels B-D show the corresponding performance statistics for the PRCC modified risk-based portfolios. The baseline implementation in Panel B uses volatility as risk measure in the PRCC and relative performance, and sets $\zeta = 10\%$. Panel C reports to results when $\zeta = 5\%$. We do not report the results for the minimum variance and minimum downside risk portfolio, since the results are identical as in Panel B. Panel D uses the modified VaR as risk measure in the PRCC and relative performance. Panel E reports results for the maximum Sharpe ratio portfolio. The performance measures are the same as defined in Table 3. We also report the values of the PRCC using volatility as the risk measure (PRCC[†]) and mVaR as the risk measure (PRCC[‡]). The last column reports the tracking error (TE) of the PRCC modified portfolio weights compared with the corresponding risk-based benchmark. The sign <0.01 indicates when the number is between 0 and 0.01%. Except for the SR and mSR, all numbers are reported in percentage points. The sample period ranges from January 1988 to August 2015 for a total of 332 monthly observations.

	Performance measures							TE
	Mean	Sd	SR	PRCC [†]	mVaR	mSR	PRCC [‡]	
<i>Panel A: Risk-based portfolios</i>								
Min downside risk	2.72	3.01	0.26	< 0.01	0.96	0.24	< 0.01	
Min variance	2.66	3.01	0.26	< 0.01	0.96	0.23	< 0.01	
Inverse volatility weighted	3.82	4.89	0.23	0.08	1.97	0.16	0.16	
Equally-weighted	5.02	8.92	0.16	0.08	3.99	0.10	0.11	
Equal-risk-contribution	3.58	4.55	0.23	0.09	1.80	0.17	0.18	
Max diversification	3.14	3.96	0.23	0.11	1.51	0.17	0.25	
<i>Panel B: PRCC modified risk-based portfolios (volatility and $\zeta = 10\%$)</i>								
Min downside risk	2.88	3.19	0.26	< 0.01	1.09	0.22	0.02	3.61
Min variance	2.75	3.11	0.26	< 0.01	1.01	0.23	< 0.01	3.10
Inverse volatility weighted	3.77	4.83	0.23	0.04	1.92	0.16	0.07	10.00
Equally-weighted	5.86	10.43	0.16	0.04	4.66	0.10	0.07	7.59
Equal-risk-contribution	3.65	4.65	0.23	0.04	1.83	0.17	0.09	10.00
Max diversification	2.51	3.16	0.23	< 0.01	1.02	0.21	< 0.01	10.00
<i>Panel C: PRCC modified risk-based portfolios (volatility and $\zeta = 5\%$)</i>								
Inverse volatility weighted	3.64	4.67	0.23	0.05	1.85	0.16	0.10	5.00
Equally-weighted	5.54	9.87	0.16	0.05	4.41	0.10	0.07	5.00
Equal-risk-contribution	3.51	4.47	0.23	0.06	1.75	0.17	0.12	5.00
Max diversification	2.61	3.28	0.23	0.04	1.11	0.20	0.07	5.00
<i>Panel D: PRCC modified risk-based portfolios (mVaR and $\zeta = 10\%$)</i>								
Min downside risk	2.89	3.12	0.27	< 0.01	1.01	0.24	< 0.01	3.78
Min variance	2.84	3.12	0.26	< 0.01	1.02	0.23	< 0.01	3.85
Inverse volatility weighted	3.77	4.88	0.22	0.04	1.94	0.16	0.08	10.00
Equally-weighted	5.81	10.41	0.16	0.04	4.62	0.10	0.06	8.98
Equal-risk-contribution	3.64	4.66	0.23	0.04	1.84	0.17	0.09	10.00
Max diversification	3.28	4.11	0.23	0.06	1.58	0.17	0.13	10.00
<i>Panel E: Other benchmark portfolios</i>								
Max Sharpe ratio	3.19	3.30	0.28	0	1.08	0.25	0.02	
Max modified Sharpe ratio	2.99	3.14	0.28	< 0.01	0.98	0.25	0	

Table 5: In-sample attribution analysis

This table presents the results of the in-sample attribution analysis of annualized PRCC for traditional benchmarks (left-part) and the PRCC modified counter-parts (right-part). For each strategy, the table reports the optimized weight vector \mathbf{w}^* , the contributions to the annualized excess portfolio return ($C_i^{\text{Mean}} \equiv 12w_i^* \tilde{\mu}_i$), the annualized portfolio standard deviation ($C_i^{\text{Sd}} \equiv \sqrt{12}w_i[\mathbf{w}^* \boldsymbol{\Sigma}]_i / \sigma_p(\mathbf{w}^*)$), and the component performance/risk contribution ($CPRC_i \equiv C_i^{\text{Mean}} - (\sum_{i=1}^n C_i^{\text{Mean}} / \sum_{i=1}^n C_i^{\text{Sd}}) C_i^{\text{Sd}}$). All reported numbers are expressed in percentage points. The six asset classes considered are the MSCI World index-developed countries (Eq-DE), the MSCI Emerging markets index (Eq-EM), the US Government bond index (Bo-GO), the US corporate bond master index (Bo-CO), NAREIT and the Gold spot index (Gold). The in-sample period ranges from January 1988 to August 2015 for a total of 332 monthly observations.

	Traditional benchmark portfolios						PRCC modified benchmark portfolios					
	Eq-DE	Eq-EM	Bo-GO	Bo-CO	NAREIT	Gold	Eq-DE	Eq-EM	Bo-GO	Bo-CO	NAREIT	Gold
<i>Panel A: Minimum downside risk portfolio</i>												
\mathbf{w}^*	3.16	2.30	93.10	0	0.59	0.86	5.04	0	86.99	0	6.19	1.78
C_i^{Mean}	0.16	0.22	2.28	0	0.04	< 0.01	0.26	0	2.13	0	0.47	0.02
C_i^{Sd}	0.10	0.09	2.77	0	0.02	0.02	0.26	0	2.35	0	0.52	0.06
$CPRC_i$	0.07	0.14	-0.23	0	0.03	-0.01	0.02	0	< 0.01	0	< 0.01	-0.03
<i>Panel B: Minimum variance portfolio</i>												
\mathbf{w}^*	3.77	1.66	93.25	0	0.07	1.25	9.63	0.08	89.01	1.14	0.14	0
C_i^{Mean}	0.20	0.16	2.28	0	< 0.01	0.01	0.50	0.01	2.18	0.05	0.01	0
C_i^{Sd}	0.11	0.05	2.81	0	< 0.01	0.04	0.57	< 0.01	2.47	0.05	< 0.01	0
$CPRC_i$	0.10	0.12	-0.20	0	< 0.01	-0.02	-0.01	< 0.01	-0.01	< 0.01	< 0.01	0
<i>Panel C: Inverse volatility weighted portfolio</i>												
\mathbf{w}^*	9.05	5.80	42.75	26.03	7.74	8.62	9.49	0	28.51	45.06	8.80	8.14
C_i^{Mean}	0.47	0.57	1.05	1.05	0.58	0.09	0.49	0	0.70	1.83	0.66	0.09
C_i^{Sd}	0.94	0.90	0.58	1.01	0.90	0.57	0.86	0	0.47	1.99	1.03	0.49
$CPRC_i$	-0.26	-0.13	0.60	0.27	-0.12	-0.35	-0.17	0	0.33	0.27	-0.13	-0.30
<i>Panel D: Equally-weighted portfolio</i>												
\mathbf{w}^*	16.67	16.67	16.67	16.67	16.67	16.67	9.86	19.89	14.81	12.67	31.99	10.77
C_i^{Mean}	0.87	1.63	0.41	0.68	1.26	0.18	0.51	1.94	0.36	0.51	2.42	0.11
C_i^{Sd}	2.01	3.27	0.04	0.41	2.11	1.09	1.14	3.79	0.01	0.30	4.71	0.48
$CPRC_i$	-0.27	-0.21	0.39	0.44	0.08	-0.43	-0.13	-0.19	0.36	0.35	-0.23	-0.16
<i>Panel E: Equal-risk-contribution portfolio</i>												
\mathbf{w}^*	7.84	5.22	50.36	19.65	6.93	10.01	9.39	0	35.37	38.07	8.84	8.34
C_i^{Mean}	0.41	0.51	1.23	0.80	0.52	0.11	0.49	0	0.87	1.54	0.67	0.09
C_i^{Sd}	0.76	0.76	0.76	0.76	0.76	0.76	0.84	0	0.60	1.66	1.03	0.53
$CPRC_i$	-0.19	-0.09	0.64	0.20	-0.07	-0.49	-0.17	0	0.40	0.24	-0.14	-0.33
<i>Panel F: Maximum diversification portfolio</i>												
\mathbf{w}^*	6.01	4.89	72.05	0	6.28	10.77	0.23	0.04	92.38	4.77	0.15	2.43
C_i^{Mean}	0.31	0.48	1.76	0	0.47	0.11	0.01	< 0.01	2.26	0.19	0.01	0.03
C_i^{Sd}	0.50	0.63	1.27	0	0.61	0.94	< 0.01	< 0.01	2.89	0.18	< 0.01	0.09
$CPRC_i$	-0.08	-0.03	0.76	0	-0.01	-0.63	0.01	< 0.01	-0.03	0.05	0.01	-0.04

Table 6: Out-of-sample performance results

This table presents the out-of-sample performance results for the risk-based portfolios, their PRCC modification, and the maximum (modified) Sharpe ratio portfolio. All reported statistics are as defined in Tables 3–4. The out-of-sample period ranges from January 1991 to August 2015, for a total of 296 monthly observations.

	\$	GR	Mean	Sd	SR	Sk	Ku	MDD	mVaR	mSR	TE
<i>Panel A: Risk-based portfolios</i>											
Min downside risk	3.73	5.48	2.64	3.03	0.25	-0.68	2.60	4.86	1.10	0.20	
Min variance	3.56	5.29	2.45	2.95	0.24	-0.51	1.75	4.57	1.05	0.19	
Inverse volatility weighted	5.40	7.07	4.22	4.94	0.25	-1.25	5.82	14.76	2.06	0.17	
Equally-weighted	7.01	8.22	5.59	9.12	0.18	-1.18	6.79	31.64	4.08	0.11	
Equal-risk-contribution	4.44	6.23	3.48	5.85	0.17	-3.29	29.54	24.62	2.48	0.12	
Max diversification	4.15	5.93	3.10	3.99	0.22	-0.68	2.08	7.93	1.57	0.16	
<i>Panel B: PRCC modified risk-based portfolios with volatility as risk measure; $\zeta = 10\%$</i>											
Min downside risk	4.12	5.91	3.06	3.49	0.25	-0.62	2.35	5.84	1.29	0.20	5.34
Min variance	3.76	5.52	2.68	3.30	0.23	-0.57	1.41	3.99	1.24	0.18	6.07
Inverse volatility weighted	5.67	7.29	4.43	5.10	0.25	-1.37	10.76	17.48	2.02	0.18	9.33
Equally-weighted	7.94	8.76	6.01	8.17	0.21	-0.83	4.65	27.95	3.44	0.14	9.98
Equal-risk-contribution	4.99	6.74	3.96	5.96	0.19	-3.28	29.07	28.28	2.51	0.13	9.52
Max diversification	4.46	6.25	3.41	4.16	0.24	-0.08	0.91	5.79	1.46	0.19	8.06
<i>Panel C: PRCC modified risk-based portfolios with volatility as risk measure; $\zeta = 5\%$</i>											
Min downside risk	3.95	5.73	2.88	3.29	0.25	-0.75	3.10	5.16	1.22	0.20	3.51
Min variance	3.62	5.35	2.52	3.12	0.23	-0.51	1.32	3.85	1.14	0.18	3.60
Inverse volatility weighted	5.30	6.99	4.14	4.96	0.24	-1.14	6.35	16.97	2.02	0.17	4.96
Equally-weighted	7.34	8.42	5.72	8.54	0.19	-1.09	6.06	30.19	3.75	0.13	5.00
Equal-risk-contribution	4.57	6.35	3.60	5.84	0.18	-3.42	30.41	27.34	2.47	0.12	5.00
Max diversification	4.26	6.05	3.21	3.93	0.24	-0.39	0.82	7.04	1.47	0.18	4.59
<i>Panel D: PRCC modified risk-based portfolios with mVaR as risk measure; $\zeta = 10\%$</i>											
Min downside risk	4.05	5.83	2.98	3.34	0.26	-0.39	1.71	5.06	1.17	0.21	4.90
Min variance	3.86	5.62	2.79	3.41	0.24	-0.62	2.31	4.72	1.27	0.18	6.04
Inverse volatility weighted	5.83	7.41	4.52	4.73	0.28	-0.09	1.44	12.26	1.63	0.23	9.25
Equally-weighted	7.56	8.55	5.81	8.22	0.20	-0.73	4.01	28.69	3.46	0.14	9.85
Equal-risk-contribution	5.04	6.78	4.01	5.99	0.19	-3.25	28.35	28.82	2.54	0.13	9.42
Max diversification	4.48	6.27	3.42	4.08	0.24	-0.40	1.42	5.62	1.51	0.19	7.80
<i>Panel E: Other benchmark portfolios</i>											
Max Sharpe ratio	5.51	7.16	4.36	6.04	0.21	-0.36	2.63	12.72	2.35	0.15	
Max modified Sharpe ratio	5.01	6.75	3.92	4.97	0.23	-0.38	2.21	10.28	1.89	0.17	

Figure 1: Illustration of the PRCC diagram

The figure presents the performance contribution (vertical axis) and risk contribution (horizontal axis) of four assets (the four dots labeled by A1-A4) in a stylized portfolio. These assets have performance contributions of 1.4, 1.6, 1.7 and 2.1. Their risk contributions are 1, 2, 2.5 and 3. The slope of the line corresponds to the relative performance ratio of the portfolio which is equal to 0.8. The PRCC measure is 0.14.

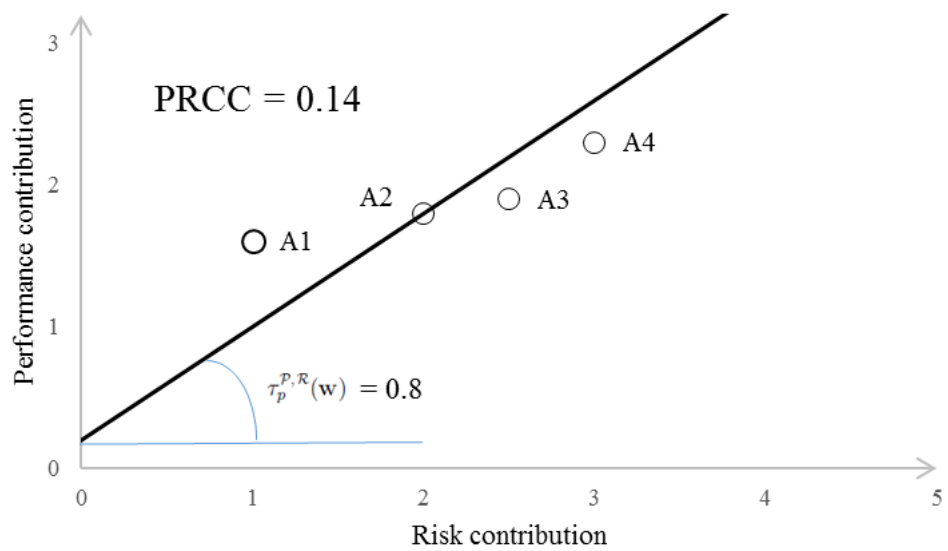
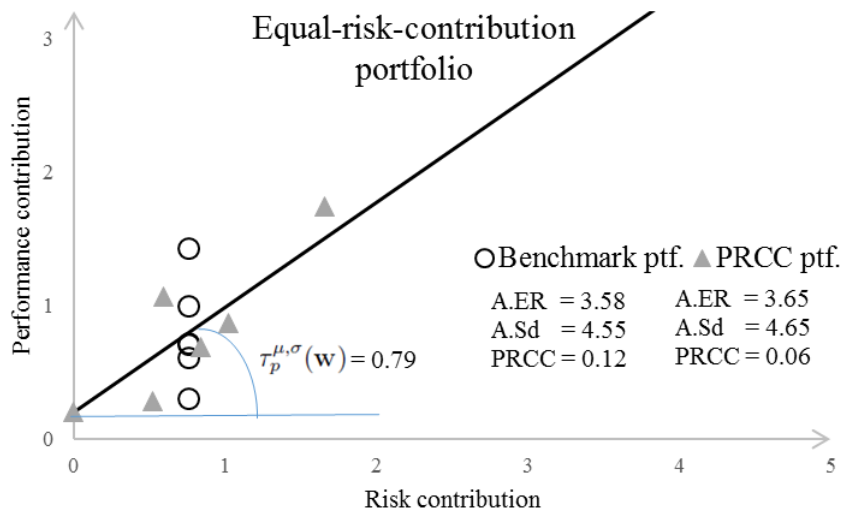
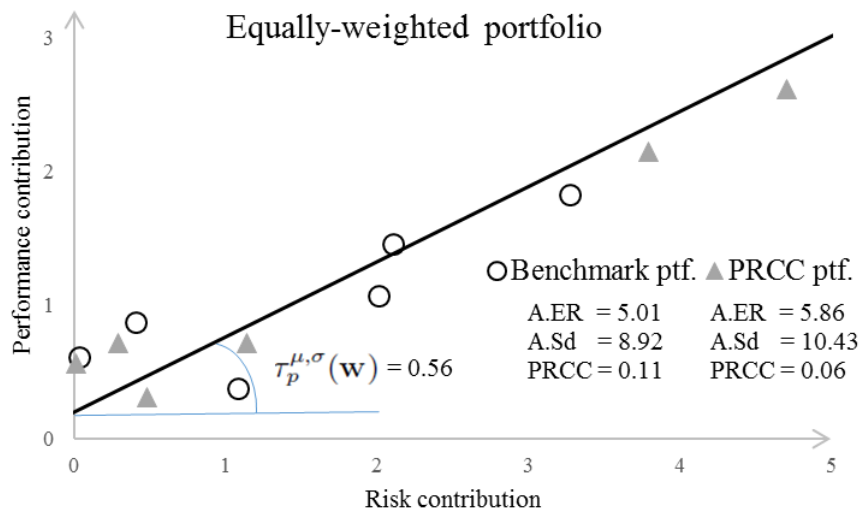


Figure 2: PRCC-based weight modification in case of the equally-weighted and equal-risk-contribution portfolios

The plots present performance contributions (vertical axis) and risk contributions (horizontal axis) of assets in risk-based portfolios (*i.e.*, Benchmark ptf.) and their PRCC modified portfolios (*i.e.*, PRCC ptf.) of two representative portfolios (the equally-weighted and equal-risk-contribution portfolios) reported in Table 5. The slope of the line corresponds to the Sharpe ratio of the portfolio.



Appendix A. Simplification of the PRCC formula

Let $a_i \equiv C_i^{\mathcal{P}}(\mathbf{w}) - \tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w})C_i^{\mathcal{R}}(\mathbf{w})$. The PRCC is proportional to the sum of squared differences between couples in $\{a_1, \dots, a_N\}$. The simplification uses that $\sum_{i=1}^N a_i = 0$, implying that:

$$\left(\sum_{i=1}^N a_i \right)^2 = \sum_{i=1}^N a_i^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N a_i a_j = 0,$$

and thus:

$$2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N a_i a_j = - \sum_{i=1}^N a_i^2. \quad (\text{A.1})$$

It follows that:

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N (a_i - a_j)^2 &= 2 \left[\sum_{i=1}^{N-1} \sum_{j=i+1}^N (a_i - a_j)^2 \right] \\ &= 2 \left[(N-1) \sum_{i=1}^N a_i^2 - 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N a_i a_j \right] \\ &= 2 \left[(N-1) \sum_{i=1}^N a_i^2 + \sum_{i=1}^N a_i^2 \right] \\ &= 2N \sum_{i=1}^N a_i^2, \end{aligned}$$

where we use (A.1) in the last equality.

Appendix B. PRCC of the maximum relative performance portfolio

The maximum relative performance portfolio maximizes $\tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w}) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})}$ under a full investment constraint. The corresponding Lagrangian is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \frac{\mathcal{P}_p(\mathbf{w})}{\mathcal{R}_p(\mathbf{w})} - l(\mathbf{w}'\mathbf{1} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, the portfolio weights need to be such that:

$$\begin{aligned} \partial_i \mathcal{L}(\mathbf{w}^*, l) &= \frac{1}{\mathcal{R}_p^2(\mathbf{w}^*)} [\mathcal{R}_p(\mathbf{w}^*) \partial_i \mathcal{P}_p(\mathbf{w}^*) - \mathcal{P}_p(\mathbf{w}^*) \partial_i \mathcal{R}_p(\mathbf{w}^*)] - l \\ &= \frac{1}{\mathcal{R}_p(\mathbf{w}^*)} [\partial_i \mathcal{P}_p(\mathbf{w}^*) - \tau_p^{\mathcal{P}, \mathcal{R}}(\mathbf{w}^*) \partial_i \mathcal{R}_p(\mathbf{w}^*)] - l = 0. \end{aligned}$$

Multiplying both sides by w_i^* , we have $\frac{1}{\mathcal{R}_p(\mathbf{w}^*)} \left[\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) \mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) \right] - l w_i^* = 0$, and thus:

$$\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) \mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) = \mathcal{R}_p(\mathbf{w}^*) l w_i^*. \quad (\text{B.1})$$

Since $\sum_{i=1}^N \left[\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) \mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) \right] = 0$, it follows from (B.1) that $\mathcal{R}_p(\mathbf{w}^*) l \mathbf{1}' \mathbf{w}^* = 0$. Under a full investment constraint, $\mathbf{1}' \mathbf{w}^* = 1$ and $\mathcal{R}_p(\mathbf{w}^*) > 0$, therefore $l = 0$. Combining this with (B.1), we obtain:

$$\mathcal{C}_i^{\mathcal{P}}(\mathbf{w}^*) - \tau_p^{\mathcal{P},\mathcal{R}}(\mathbf{w}^*) \mathcal{C}_i^{\mathcal{R}}(\mathbf{w}^*) = 0,$$

for all i . The PRCC measure is thus zero for the maximum relative performance portfolio.

Appendix C. Volatility-based PRCC for risk-based portfolios

Minimum downside risk portfolio. As in Boudt et al. (2008) and Roncalli (2015), we assume here a downside risk measure that can be written as:

$$\mathcal{R}_p(\mathbf{w}) \equiv -\mathbf{w}' \boldsymbol{\mu} + c_Z \sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}},$$

with c_Z a constant. The Lagrangian corresponding to the minimum downside risk portfolio under a full investment constraint is:

$$\mathcal{L}(\mathbf{w}, l) \equiv -\mathbf{w}' \boldsymbol{\mu} + c_Z \sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}} - l(\mathbf{w}' \mathbf{1} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions the portfolio weights need to be such that:

$$-\mu_i + c_Z \frac{[\boldsymbol{\Sigma} \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = l. \quad (\text{C.1})$$

Multiplying both sides by w_i^* and taking the sum, we get:

$$\sum_{i=1}^N w_i^* \left[-\mu_i + c_Z \frac{[\boldsymbol{\Sigma} \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} \right] = l \sum_{i=1}^N w_i^* = l,$$

because of the full investment constraint. In vector notation, this is equivalent to $l = -\boldsymbol{\mu}' \mathbf{w}^* + c_Z \frac{\sigma_p^2(\mathbf{w}^*)}{\sigma_p(\mathbf{w}^*)}$. Since $\tilde{\boldsymbol{\mu}} \equiv \boldsymbol{\mu} - r_f$, where r_f is the risk-free rate, we obtain $l = -\tilde{\boldsymbol{\mu}}' \mathbf{w}^* - r_f + c_Z \sigma_p(\mathbf{w}^*) = \sigma_p(\mathbf{w}^*) \left[-\tau_p^{\mu,\sigma}(\mathbf{w}^*) + c_Z \right] - r_f$. Combining this with (C.1), we have:

$$\mu_i = c_Z \frac{[\boldsymbol{\Sigma} \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} + \sigma_p(\mathbf{w}^*) \left[\tau_p^{\mu,\sigma}(\mathbf{w}^*) - c_Z \right] + r_f.$$

Equivalently, $\tilde{\mu}_i = c_Z \frac{[\Sigma \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} + \sigma_p(\mathbf{w}^*) [\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z]$. Hence, the Performance/Risk contribution of asset i is:

$$\begin{aligned} C_i^\mu(\mathbf{w}^*) - \tau_p^{\mu, \sigma}(\mathbf{w}^*) C_i^\sigma(\mathbf{w}^*) &= w_i^* \left\{ c_Z \frac{[\Sigma \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} + \sigma_p(\mathbf{w}^*) [\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z] - \tau_p^{\mu, \sigma}(\mathbf{w}^*) \frac{[\Sigma \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} \right\} \\ &= [\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z] [w_i^* \sigma_p(\mathbf{w}^*) - C_i^\sigma(\mathbf{w}^*)] \\ &= [\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z] \cdot \frac{1}{\sigma_p(\mathbf{w}^*)} \cdot w_i^* [\sigma_p^2(\mathbf{w}^*) - [\Sigma \mathbf{w}^*]_i]. \end{aligned}$$

The PRCC measure in (8) is thus:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \frac{[\tau_p^{\mu, \sigma}(\mathbf{w}^*) - c_Z]^2}{\sigma_p^2(\mathbf{w}^*)} \cdot \sum_{i=1}^N \left\{ w_i^* [\sigma_p^2(\mathbf{w}^*) - [\Sigma \mathbf{w}^*]_i] \right\}^2.$$

Minimum variance portfolio. The minimum variance portfolio minimizes the portfolio variance $\sigma_p^2(\mathbf{w}) = \mathbf{w}' \Sigma \mathbf{w}$ under the full investment constraint $\mathbf{w}' \boldsymbol{\iota} = 1$. The corresponding Lagrangian is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \mathbf{w}' \Sigma \mathbf{w} - l(\mathbf{w}' \boldsymbol{\iota} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, it follows that $\Sigma \mathbf{w}^* = \frac{1}{2} l \boldsymbol{\iota}$. Since $\sigma_p^2(\mathbf{w}^*) = (\mathbf{w}^*)' \Sigma \mathbf{w}^* = \frac{1}{2} l \boldsymbol{\iota}' \mathbf{w}^*$ and because of the full investment constraint $\boldsymbol{\iota}' \mathbf{w}^* = 1$, it follows that $\frac{1}{2} l = \sigma_p^2(\mathbf{w}^*)$ and thus $\Sigma \mathbf{w}^* = \sigma_p^2(\mathbf{w}^*) \boldsymbol{\iota}$. Hence the risk contribution of asset i is $C_i^\sigma \equiv w_i^* \frac{[\Sigma \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = w_i^* \sigma_p(\mathbf{w}^*)$. Using this result, we can thus rewrite PRCC in (8) as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \sum_{i=1}^N [C_i^\mu(\mathbf{w}^*) - \tau_p^{\mu, \sigma}(\mathbf{w}^*) w_i^* \sigma_p(\mathbf{w}^*)]^2 = \frac{1}{N} \sum_{i=1}^N \left\{ w_i^* [\tilde{\mu}_i - \tilde{\mu}_p(\mathbf{w}^*)] \right\}^2,$$

since $\tau_p^{\mu, \sigma}(\mathbf{w}^*) \equiv \tilde{\mu}_p(\mathbf{w}^*) / \sigma_p(\mathbf{w}^*)$.

Inverse volatility weighted portfolio. Let us define $\xi_i \equiv 1/\sigma_i$ and $\boldsymbol{\xi} \equiv (\xi_1, \dots, \xi_N)'$. Then, the weights of the inverse volatility weighted portfolio are given by $\mathbf{w} \equiv \boldsymbol{\xi} / \boldsymbol{\xi}' \boldsymbol{\iota}$. The covariance matrix can be decomposed as $\Sigma \equiv \mathbf{D} \mathbf{R} \mathbf{D}$, where \mathbf{D} is a diagonal matrix containing the variances $(\sigma_1, \dots, \sigma_N)'$ and \mathbf{R} is the correlation matrix. Using $\mathbf{D} \boldsymbol{\xi} = \boldsymbol{\iota}$, the portfolio volatility is:

$$\sigma_p(\mathbf{w}^*) \equiv \sqrt{(\mathbf{w}^*)' \Sigma \mathbf{w}^*} = \sqrt{\frac{\boldsymbol{\xi}' \mathbf{D} \mathbf{R} \mathbf{D} \boldsymbol{\xi}}{\boldsymbol{\xi}' \boldsymbol{\iota}}} = \frac{1}{\boldsymbol{\xi}' \boldsymbol{\iota}} \sqrt{\boldsymbol{\iota}' \mathbf{R} \boldsymbol{\iota}}.$$

Component risk contribution of asset i is then:

$$C_i^\sigma \equiv w_i^* \frac{[\Sigma \mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{1}{\boldsymbol{\xi}' \boldsymbol{\iota}} \frac{1}{\sigma_i} \frac{[\mathbf{D} \mathbf{R} \boldsymbol{\xi}]_i}{\sqrt{\boldsymbol{\iota}' \mathbf{R} \boldsymbol{\iota}}} = \frac{1}{\boldsymbol{\xi}' \boldsymbol{\iota}} \frac{[\mathbf{R} \boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}' \mathbf{R} \boldsymbol{\iota}}}.$$

Then, the PRCC of the inverse volatility weighted portfolio can be rewritten as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \frac{1}{(\boldsymbol{\xi}'\boldsymbol{\iota})^2} \cdot \sum_{i=1}^N \left[\frac{\tilde{\mu}_i}{\sigma_i} - \tau_p(\mathbf{w}^*) \frac{[\mathbf{R}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\mathbf{R}\boldsymbol{\iota}}} \right]^2.$$

Equally-weighted portfolio. For the equally-weighted portfolio $\mathbf{w}^* \equiv \frac{\boldsymbol{\iota}}{N}$, the portfolio risk is:

$$\sigma_p(\mathbf{w}^*) \equiv \sqrt{(\mathbf{w}^*)'\boldsymbol{\Sigma}\mathbf{w}^*} = \sqrt{\frac{1}{N}\boldsymbol{\iota}'\boldsymbol{\Sigma}\frac{1}{N}\boldsymbol{\iota}} = \frac{1}{N}\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}.$$

The component risk contribution of asset i is:

$$C_i^\sigma \equiv w_i^* \frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{1}{N} \frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}}.$$

Then, the PRCC of the equally-weighted portfolio can be rewritten as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \sum_{i=1}^N \left[\frac{1}{N}\tilde{\mu}_i - \tau_p^{\mu,\sigma}(\mathbf{w}^*) \frac{1}{N} \frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}} \right]^2 = \frac{1}{N^3} \cdot \sum_{i=1}^N \left[\tilde{\mu}_i - \tau_p^{\mu,\sigma}(\mathbf{w}^*) \frac{[\boldsymbol{\Sigma}\boldsymbol{\iota}]_i}{\sqrt{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\iota}}} \right]^2.$$

Equal-risk-contribution portfolio. The equal-risk-contribution portfolio aims at equalizing the component-risk-contributions: $C_i^\sigma(\mathbf{w}^*) = C_j^\sigma(\mathbf{w}^*) \forall i, j$. As $\sum_{i=1}^N C_i^\sigma(\mathbf{w}^*) = \sigma_p(\mathbf{w}^*)$, it follows that $C_i^\sigma(\mathbf{w}^*) = \frac{\sigma_p(\mathbf{w}^*)}{N}$. Hence, the component performance/risk contribution of asset i is:

$$\text{CPRC}_i(\mathbf{w}^*) = C_i^\mu(\mathbf{w}^*) - \tau_p^{\mu,\sigma}(\mathbf{w}^*) \frac{\sigma_p(\mathbf{w}^*)}{N} = w_i^* \tilde{\mu}_i - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N}.$$

So the PRCC measure in (8) can be rewritten as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \sum_{i=1}^N \left[w_i^* \tilde{\mu}_i - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{N} \right]^2.$$

Maximum diversification portfolio. The maximum diversification portfolio maximize the diversification ratio $\mathbf{w}'\boldsymbol{\sigma}/\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$, where $\boldsymbol{\sigma}$ is the vector of volatilities. The corresponding Lagrangian under a full investment constraint is:

$$\mathcal{L}(\mathbf{w}, l) \equiv \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} - l(\mathbf{w}'\boldsymbol{\iota} - 1),$$

with $l \in \mathbb{R}$. From the first-order conditions, it follows that:

$$\frac{\sigma_p(\mathbf{w}^*)\sigma_i - \mathbf{w}^{*\prime} \boldsymbol{\sigma} \frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)}}{\sigma_p^2(\mathbf{w}^*)} = l. \quad (\text{C.2})$$

Multiplying both sides by w_i^* and taking the sum, we get:

$$\frac{\sigma_p(\mathbf{w}^*)\boldsymbol{\sigma}'\mathbf{w}^* - \boldsymbol{\sigma}'\mathbf{w}^*\sigma_p(\mathbf{w}^*)}{\sigma_p^2(\mathbf{w}^*)} = l \sum_{i=1}^N w_i^* = l.$$

Since the left-hand side of the equation is zero and because of the full investment constraint $\iota'\mathbf{w}^* = 1$, it follows that $l = 0$ and thus, given (C.2) we obtain:

$$\sigma_p(\mathbf{w}^*)\sigma_i - \boldsymbol{\sigma}'\mathbf{w}^* \frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = 0.$$

Equivalently, $\frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} = \frac{\sigma_p(\mathbf{w}^*)\sigma_i}{\boldsymbol{\sigma}'\mathbf{w}^*}$. Using this result, we can rewrite PRCC in (8) as:

$$\text{PRCC}(\mathbf{w}^*) = \frac{1}{N} \sum_{i=1}^N \left[w_i^* \left(\tilde{\mu}_i - \tau_p^{\mu, \sigma}(\mathbf{w}^*) \frac{[\boldsymbol{\Sigma}\mathbf{w}^*]_i}{\sigma_p(\mathbf{w}^*)} \right) \right]^2 = \frac{1}{N} \sum_{i=1}^N \left[w_i^* \sigma_i \left(\frac{\tilde{\mu}_i}{\sigma_i} - \frac{\tilde{\boldsymbol{\mu}}'\mathbf{w}^*}{\boldsymbol{\sigma}'\mathbf{w}^*} \right) \right]^2.$$

For the maximum diversification portfolio, the PRCC measure is thus zero when all assets have the same Sharpe ratio. Indeed, when σ_i is proportional to μ_i , maximizing the diversification ratio is equivalent to maximizing the portfolio's Sharpe ratio.

Appendix D. Modified VaR and ES

Let us denote the co-skewness matrix as:

$$M_3 \equiv \mathbb{E} [(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})' \otimes (\mathbf{r} - \boldsymbol{\mu})'] ,$$

and co-kurtosis matrix:

$$M_4 \equiv \mathbb{E} [(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})' \otimes (\mathbf{r} - \boldsymbol{\mu})' \otimes (\mathbf{r} - \boldsymbol{\mu})'] ,$$

where $\mathbf{r} \equiv (r_1, \dots, r_N)'$, $\boldsymbol{\mu} \equiv (\mu_1, \dots, \mu_N)'$ and where \otimes stands for the Kronecker product (see, e.g., Jondeau and Rockinger, 2006). Define $m_q \equiv \mathbb{E} [(r_p - \mathbf{w}'\boldsymbol{\mu})^q]$ as the q -th centered portfolio moment, then $\partial_i m_q$ is its partial derivative with respect to w_i . We have that:

$$\begin{aligned} m_2 &= \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} & \partial_i m_2 &= 2[\boldsymbol{\Sigma}\mathbf{w}]_i \\ m_3 &= \mathbf{w}'M_3(\mathbf{w} \otimes \mathbf{w}) & \partial_i m_3 &= 3[M_3(\mathbf{w} \otimes \mathbf{w})]_i \\ m_4 &= \mathbf{w}'M_4(\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}) & \partial_i m_4 &= 4[M_4(\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w})]_i. \end{aligned}$$

The portfolio skewness $s_p(\mathbf{w})$ and excess kurtosis $k_p(\mathbf{w})$ and their partial derivatives are then given by:

$$\begin{aligned} s_p(\mathbf{w}) &\equiv m_3/m_2^{3/2} & \partial_i s_p(\mathbf{w}) &= (2m_2^{3/2} \partial_i m_3 - 3m_3 m_2^{1/2} \partial_i m_2)/2m_2^3 \\ k_p(\mathbf{w}) &\equiv m_4/m_2^2 - 3 & \partial_i k_p(\mathbf{w}) &= (m_2 \partial_i m_4 - 2m_4 \partial_i m_2)/m_2^3. \end{aligned}$$

Let $z_\alpha \equiv \Phi^{-1}(\alpha)$, then define:

$$g_\alpha(\mathbf{w}) \equiv z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)s_p(\mathbf{w}) + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)k_p(\mathbf{w}) - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)s_p^2(\mathbf{w}).$$

We have:

$$\partial_i g_\alpha(\mathbf{w}) = \frac{1}{6}(z_\alpha^2 - 1)\partial_i s_p(\mathbf{w}) + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)\partial_i k_p(\mathbf{w}) - \frac{1}{18}(2z_\alpha^3 - 5z_\alpha)s_p(\mathbf{w})\partial_i s_p(\mathbf{w}).$$

According to Zangari (1996) and Martin and Arora (2015), the modified VaR is then defined as:

$$\text{mVaR}_\alpha(\mathbf{w}) \equiv -\mathbf{w}'\boldsymbol{\mu} - \sigma(\mathbf{w})g_\alpha(\mathbf{w}).$$

It is straightforward to calculate the marginal contribution of asset i in mVaR_α :

$$\partial_i \text{mVaR}_\alpha(\mathbf{w}) = -\mu_i - \frac{[\boldsymbol{\Sigma}\mathbf{w}]_i}{\sigma_p(\mathbf{w})}g_\alpha(\mathbf{w}) - \sigma_p(\mathbf{w})\partial_i g_\alpha(\mathbf{w}).$$

The modified ES was introduced in Boudt et al. (2008). We use the equivalent notation of Martin and Arora (2015):

$$\begin{aligned} \text{mES}_\alpha(\mathbf{w}) &\equiv -\mu_p(\mathbf{w}) \\ &+ \frac{1}{\alpha}\phi(g_\alpha(\mathbf{w})) \left[1 + \frac{1}{6}g_\alpha^3(\mathbf{w})s_p(\mathbf{w}) + \frac{1}{72}(g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3)s_p^2(\mathbf{w}) \right. \\ &\left. + \frac{1}{24}(g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1)k_p(\mathbf{w}) \right] \sigma_p(\mathbf{w}). \end{aligned}$$

Applying the property $\phi'(z) = -z\phi(z)$, we obtain:

$$\begin{aligned}
\partial_i \text{mES}_\alpha(\mathbf{w}) &= -\mu_i \\
&+ \frac{1}{\alpha} \left[\phi(g_\alpha(\mathbf{w})) \frac{[\Sigma \mathbf{w}]_i}{\sigma_p(\mathbf{w})} - \sigma_p(\mathbf{w}) g_\alpha \phi(g_\alpha(\mathbf{w})) \partial_i g_\alpha(\mathbf{w}) \right] \left[1 + \frac{1}{6} g_\alpha^3(\mathbf{w}) s_p(\mathbf{w}) \right. \\
&+ \frac{1}{72} (g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3) s_p^2(\mathbf{w}) + \frac{1}{24} (g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1) k_p(\mathbf{w}) \left. \right] \\
&+ \frac{1}{\alpha} \phi(g_\alpha) \sigma(\mathbf{w}) \left[\frac{1}{2} g_\alpha^2 \partial_i g_\alpha s_p(\mathbf{w}) + \frac{1}{6} g_\alpha^3 \partial_i s_p(\mathbf{w}) \right. \\
&+ \frac{1}{36} (g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3) s_p(\mathbf{w}) \partial_i s_p(\mathbf{w}) \\
&+ \frac{1}{12} (g_\alpha^5(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) - 6g_\alpha^3(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) + 3g_\alpha(\mathbf{w}) \partial_i g_\alpha(\mathbf{w})) s_p^2(\mathbf{w}) \\
&\left. + \frac{1}{24} (g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1) \partial_i k_p(\mathbf{w}) + \frac{1}{6} (g_\alpha^3(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) - g_\alpha(\mathbf{w}) \partial_i g_\alpha(\mathbf{w})) k_p(\mathbf{w}) \right].
\end{aligned}$$

The partial derivative of the volatility multiplier for mES in Table 2 is given by:

$$\begin{aligned}
\partial_i c_p(\mathbf{w}) &= \\
&+ \frac{1}{\alpha} [-g_\alpha \phi(g_\alpha(\mathbf{w})) \partial_i g_\alpha(\mathbf{w})] \left[1 + \frac{1}{6} g_\alpha^3(\mathbf{w}) s_p(\mathbf{w}) \right. \\
&+ \frac{1}{72} (g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3) s_p^2(\mathbf{w}) + \frac{1}{24} (g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1) k_p(\mathbf{w}) \left. \right] \\
&+ \frac{1}{\alpha} \phi(g_\alpha) \left[\frac{1}{2} g_\alpha^2 \partial_i g_\alpha s_p(\mathbf{w}) + \frac{1}{6} g_\alpha^3 \partial_i s_p(\mathbf{w}) + \frac{1}{36} (g_\alpha^6(\mathbf{w}) - 9g_\alpha^4(\mathbf{w}) + 9g_\alpha^2(\mathbf{w}) + 3) s_p(\mathbf{w}) \partial_i s_p(\mathbf{w}) \right. \\
&+ \frac{1}{12} (g_\alpha^5(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) - 6g_\alpha^3(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) + 3g_\alpha(\mathbf{w}) \partial_i g_\alpha(\mathbf{w})) s_p^2(\mathbf{w}) \\
&\left. + \frac{1}{24} (g_\alpha^4(\mathbf{w}) - 2g_\alpha^2(\mathbf{w}) - 1) \partial_i k_p(\mathbf{w}) + \frac{1}{6} (g_\alpha^3(\mathbf{w}) \partial_i g_\alpha(\mathbf{w}) - g_\alpha(\mathbf{w}) \partial_i g_\alpha(\mathbf{w})) k_p(\mathbf{w}) \right].
\end{aligned}$$

Appendix E. Historical VaR and ES estimations

We follow Harmantzis et al. (2006) to define the historical Value-at-Risk (HVaR) and the historical Expected Shortfall (HES). Let $r_p^{t|T}(\mathbf{w})$ be the portfolio return at period t (given the information set up to T). We rank the vector by the order statistics (denoted in brackets) as: $r_p^{(1)|T}(\mathbf{w}) \leq r_p^{(2)|T}(\mathbf{w}) \leq \dots \leq r_p^{(T)|T}(\mathbf{w})$. The α quantile function $F^{-1}(\alpha)$ is then estimated by:

$$F^{-1}(\alpha) \equiv r_p^{(t)|T}(\mathbf{w}), \alpha \in \left(\frac{t-1}{T}, \frac{t}{T} \right].$$

Then estimations of HVaR and HES are defined as:

$$\text{HVaR}_\alpha \equiv -F^{-1}(\alpha),$$

and:

$$\text{HES}_\alpha \equiv -\frac{1}{\lfloor \alpha T \rfloor} \sum_{t=1}^{\lfloor \alpha T \rfloor} r_p^{(t)|T}(\mathbf{w}),$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x . The component HVaR of asset i can be directly derived as $w_i r_i^{(t)|T} | r_p(\mathbf{w}) \equiv -F^{-1}(\alpha)$. Epperlein and Smillie (2006) show that such methodology suffers from the noise and the estimation using the kernel estimators is preferred. Let $K(a; b) \equiv \max(1 - |a/b|, 0)$ be the triangular kernel function. The component HVaR is then estimated as:

$$\mathcal{C}_i^{\text{HVaR}_\alpha}(\mathbf{w}) \equiv \text{HVaR}_\alpha \frac{\sum_{t=1}^T K(r_p^{t|T}(\mathbf{w}) + \text{HVaR}_\alpha; h) r_i^{t|T}}{\sum_{t=1}^T K(r_p^{t|T}(\mathbf{w}) + \text{HVaR}_\alpha; h) r_p^{t|T}(\mathbf{w})},$$

where $h \equiv 2.575\sigma_p(\mathbf{w})T^{-\frac{1}{5}}$ and $r_i^{t|T}$ is the return of asset i at time t (given the information set up to T).

We follow Yamai and Yoshida (2002) to estimate the component HES directly as $\mathbb{E}[w_i r_i | r_p(\mathbf{w}) \leq F^{-1}(\alpha)]$:

$$\mathcal{C}_i^{\text{HES}_\alpha}(\mathbf{w}) \equiv -\frac{1}{\lfloor \alpha T \rfloor} \sum_{t=1}^{\lfloor \alpha T \rfloor} w_i r_i^{(t)|T} I[w_i r_i^{(t)|T} | r_p^{(t)|T}(\mathbf{w}) \leq F^{-1}(\alpha)],$$

where $I[\cdot]$ is the indicator function.