FINVEX WHITE PAPER ON ASSET ALLOCATION WITH HIGHER ORDER MOMENTS AND FACTOR MODELS

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Executive Summary

In asset allocation, minimum variance portfolios with variance contribution constraints are nowadays popular to achieve portfolio return stability under a constraint of portfolio diversification across asset classes.

In a normal world, the variance is an appropriate risk measure and such an approach can be considered to lead to efficient results. However, in many aspects, the world of asset returns is non-normal. Its distribution tends to be asymmetric and extremes occur too often to be compatible with the tails of a normal distribution. Pure minimum variance and equal variance contribution portfolios ignore this non-normality.

We investigate the potential benefits of (i) switching from a minimum variance objective to a maximum expected utility objective that depends on the covariance, coskewness and cokurtosis, (ii) replacing an equal variance contribution constraint by an equal expected shortfall contribution constraint, and (iii) the use of statistical multifactor models for estimating the higher order comoments instead of the sample estimator or statistical single factor model.

In an application to an international diversified portfolio based on historical data, we find that accounting for the higher order moments in the portfolio objective and risk diversification constraint, increases out-of-sample returns, decreases the portfolio volatility and leads to an important reduction in the portfolio drawdown. These gains tend to be higher when using the multifactor approach to comoment estimation instead of the sample and single factor model based estimates. Further analysis will be required to confirm the robustness of these findings in other market conditions.

Introduction

The world of asset returns is non-normal. Its distribution tends to be asymmetric and extremes occur too often to be compatible with the tails of a normal distribution. An important outstanding question is whether and how to integrate this non-normality in the asset allocation decision. If there is no estimation error, most investors would be willing to sacrifice expected return and/or accept a higher volatility in exchange for a higher skewness and lower kurtosis leading to a lower downside risk (e.g. Ang, Chen and Xing, 2006; Harvey and Siddique, 2000, Scott and Horvath, 1980). This trade-off between positive preferences for odd moments (mean, skewness) and negative preferences for even moments (variance, kurtosis) can be conveniently summarized into a single objective function using a Taylor expansion of the expected utility function as objective (Martellini and Zieman, 2010).

The important caveat is that portfolio moments need to be estimated, and that the estimation error greatly amplifies when the dimension of the investment universe increases. Already for moderately large dimensions, this curse of dimensionality makes the unrestricted estimators of the first four (co)moments almost useless. Suppose e.g. that we have a universe of 20 assets, then the number of unique elements in the covariance, coskewness and cokurtosis is 210, 1540 and 8555, respectively. This is clearly an excessive number of parameters compared to the number of observations that are available in realistic applications.

We thus need to combine the data with intelligence in relation to the model that has generated the data. Martellini and Ziemann (2010) consider a single factor model. Under this approach the total number of unique elements in the covariance, coskewness and cokurtosis matrix is reduced to 83. This however induces possible specification error. Especially as far as asset allocation is concerned, it is unlikely that only one factor would be able to explain the cross-section of asset returns.

In this paper, we are among the first to derive the coskewness and cokurtosis matrix under a general multifactor model. The total number of unique elements in the covariance, coskewness and cokurtosis matrix is 112 and 151 for 2 and 3 factors, respectively. The estimation of such a number of parameters is still feasible and reduces substantially the specification bias of the single factor model.

We illustrate the usefulness of the multifactor approach to higher order comoments in an international portfolio context where the investor allocates with the purpose to maximize his expected utility. The universe consists of four equity benchmarks, nine bond indices and five commodity indices. We find that accounting for the higher order moments using a multifactor approach increases out-of-sample returns, decreases portfolio standard deviations and leads to an important reduction in the portfolio downside risk.

In what follows, we first review the methodology of factor based estimation of higher order asset comoments and the link to portfolio moments. We then study the out-of-sample performance of portfolios that use these higher order comoments. The major findings are summarized in the conclusions.

Higher order comoments and the maximum expected utility portfolio decision

The investment universe we analyse consists out of *N* assets and the investor needs to decide on the *Nx1* vector of portfolio weights *w* that optimizes a constrained objective function that depends on the first four portfolio moments. We will denote μ as the vector of expected returns on the different assets and write $m_{(k)}(w)$ as the *k*-th centered portfolio return moment:

$$m_{(k)}(w) = E[(w'(R-\mu))^k]. \tag{1}$$

Like in Martellini and Ziemann (2010) we will assume that the investor maximizes the expected value of the fourth order expansion of the Constant Relative Risk Aversion (CRRA) utility function with risk aversion parameter γ and we neutralize the effect of the mean by assuming it to be zero. This then leads to the following expected utility function:

$$EU_\gamma(w) = -\frac{\gamma}{2} m_{(2)}(w) + \frac{\gamma(\gamma+1)}{6} m_{(3)}(w) - \frac{\gamma(\gamma+1)(\gamma+2)}{24} m_{(4)}(w). \tag{2}$$

Let us take a closer look at the objective function in (2). First, note that the expected utility, ceteris paribus, decreases with the portfolio variance and kurtosis, and increases with the portfolio skewness. This reflects the general preference of investors for portfolios with low downside risk. Second, we observe that the trade-off between variance, skewness and kurtosis depends on the risk aversion parameter γ. Like in Martellini and Ziemann (2010) we will consider $\gamma = 10$ as our base case but will also try the more common choice of $\gamma = 5$. The higher the risk aversion, the more weight is put on the higher moments. Finally, note that under the assumption of normality, maximizing expected utility is equivalent to the minimum variance portfolio.

We will apply this to an asset allocation portfolio. In order to avoid that this portfolio's risk exposure is concentrated in the least risky assets, we additionally impose the (ERC) diversification constraint that all asset classes contribute equally to portfolio risk.[1] Assuming normality, the equal risk contribution constraint, implies that all assets contribute equally to the portfolio variance.

Under non-normality, the variance is no longer acceptable as a risk measure and a downside risk measure needs to be used. We will use the modified Expected Shortfall estimator proposed by Boudt et al. (2008) in which the third order Cornish-Fisher expansion is used to account for the skewness and kurtosis when estimating the portfolio risk.

[1] ERC stands for Equal Risk Contribution. The properties of ERC in variance based portfolios have been studied in Maillard et al. (2010) and Lee (2011), among others, while Boudt et al (2013) were among the first to analyze the ERC constraint using expected shortfall as the risk measure.

More precisely, the estimator for the expected shortfall at the level *α* (e.g. 5 per cent) is given by:

$$ES_{\alpha}(w) = -w'\mu + \sqrt{m_{(2)}(w)} \times \frac{1}{\alpha}\left[a_{\alpha} + b_{\alpha}k(w) + c_{\alpha}s(w) + d_{\alpha}s^{2}(w)\right], \tag{3}$$

with a_{α}, b_{α}, c_{α}, and d_{α} numbers that depend on the choice of the loss level *α* (see Boudt et al., 2008) and *s(w)* and *k(w)* are the portfolio skewness and excess kurtosis respectively:

$$\begin{aligned} s(w) &= m_{(3)}(w) / m_{(2)}^{3/2}(w), \\ k(w) &= m_{(4)}(w) / m_{(2)}^{2}(w) - 3. \end{aligned} \tag{4}$$

A Multifactor Specification of Higher Order Comoments

We have outlined a portfolio decision framework in which the first four portfolio moments determine the investment decisions. We first review the analytical expressions based on Kronecker products that facilitate the fast calculation of portfolio moments based on higher order comoment matrices. We then introduce the factor model based approach to the estimation of those higher order comoments.

General framework

We suppose to have *N* assets and wish to determine the optimal allocation to these assets based on the second, third and fourth portfolio moment. The optimal portfolio outcome will depend on all the comoments of (i) the products of two returns, i.e. the covariance of assets *i* and *j*:

$$\sigma_{i,j} = E\left[(R_{(i)} - \mu_{(i)})(R_{(j)} - \mu_{(j)})\right] \tag{5}$$

(ii) the products of three returns, i.e. the coskewness of assets *i, j* and *k*:

$$\phi_{i,j,k} = E\left[(R_{(i)} - \mu_{(i)})(R_{(j)} - \mu_{(j)})(R_{(k)} - \mu_{(k)})\right], \tag{6}$$

and (iii) the products of four returns, i.e. the cokurtosis of assets *i, j, k* and *l*:

$$\psi_{i,j,k,l} = E\left[R_{(i)} - \mu_{(i)})(R_{(j)} - \mu_{(j)})(R_{(k)} - \mu_{(k)})(R_{(l)} - \mu_{(l)})\right] \tag{7}$$

It will reveal useful to stack all these comoments into a NxN covariance matrix Σ, NxN^2 coskewness matrix Φ and NxN^3 cokurtosis matrix Ψ of the corresponding return vector *R* with mean μ_R, i.e.:

$$\begin{aligned}
\Sigma &= E\left[(R-\mu_R)(R-\mu_R)'\right] \\
\Phi &= E\left[(R-\mu_R)(R-\mu_R)'\otimes(R-\mu_R)'\right] \\
\Psi &= E\left[(R-\mu_R)(R-\mu_R)'\otimes(R-\mu_R)'\otimes(R-\mu_R)'\right],
\end{aligned} \tag{8}$$

where $\otimes$ denotes the Kronecker product.[2]

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[2] If *A* is an *mxn* matrix and *B* is a *pxq* matrix, then the Kronecker product of *A* and *B* is the matrix with *(mxp)* rows and *(nxq)* columns that stacks the direct product between each element of *A* with the matrix *B*:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}.$$

In fact, as noted e.g. by Peterson and Boudt (2008), we have the nice property that they allow us to easily calculate the *k*-th portfolio return moment:

$$
\begin{aligned}
m_{(2)}(w) &= E\left[\left(w'(R-\mu_R)\right)^2\right] = w'\Sigma w \\
m_{(3)}(w) &= E\left[\left(w'(R-\mu_R)\right)^3\right] = w'\Phi(w\otimes w) \\
m_{(4)}(w) &= E\left[\left(w'(R-\mu_R)\right)^4\right] = w'\Psi(w\otimes w\otimes w).
\end{aligned} \tag{9}
$$

The challenge is to estimate these comoments. As shown in Table 1, for moderately sized portfolios of size N=20, it requires to estimate 10,605 parameters. To avoid this curse of dimensionality, the solution we provide is to impose more structure on the data through a statistical factor model.

Table 1 Number of elements to estimate under the unrestricted and multifactor model approach

	Unrestricted	Factor model with *K* factors		
Σ	N(N+1)/2	N(K+1)+K(K+1)/2		
Φ	N(N+1)(N+2)/6	N(K+1)+ K(K+1)(K+2)/6		
Ψ	N(N+1)(N+2)(N+3)/24	N×(K+2)+ K(K+1)/2+K(K+1)(K+2)(K+3)/24		
Total	N(N+1)/2+ N(N+1)(N+2)/6 + N(N+1)(N+2)(N+3)/24	N(K+3)+[(1+(K+2)/3+(K+2)(K+3)/12]K(K+1)/2		
Total		*K=1*	*K=2*	*K=3*
N=5	120	23	37	61
N=20	10,605	83	112	151
N=100	4,598,025	403	512	631

Factor model-based decomposition of higher order comoments

A common approach is to assume that the variation in asset returns is driven by multiple common factors and idiosyncratic factors that are specific to each asset. Accounting for those factors is an important component of the asset allocation decision (e.g. Boudt and Peeters, 2013).

We will first review the factor models and then derive the expressions for the higher comoments under the multifactor model.

Review of factor models for asset returns

Three types of factor models exist. In the macroeconomic factor model, factors are observable macro-financial variables. Under the fundamental factor model, factors are created from observable asset characteristics. Finally, in statistical factor models, factors are unobservable and extracted from the asset returns.

Those three factor models can be represented in the same general form. To introduce this notation, suppose that *K* observable factors are identified as being influential for the portfolio variability, and that, at a given frequency, the asset returns $r_t=(r_{1t},\dots, r_{Nt})'$ and the factors $f_t=(f_{1t},\dots, f_{Kt})'$ are recorded.

The asset returns are assumed to depend linearly on the factors, whereby the variation in the asset returns that is not explained by the factors, is assumed to be independent of each of the factors and also to be independent across assets. In matrix notation, the system is given by:

$$r_t = a + Bf_t + e_t \,, \tag{10}$$

where $e_t=(e_{1t},\dots, e_{Nt})'$ is the $Nx1$ vector of asset specific factors and B is the NxK matrix of factor loadings (also called factor beta's) of the N assets on the K factors.

Comoments under the linear factor model

Let S be the KxK covariance matrix of the K factors, G the KxK^2 coskewness matrix of the K factors and P their KxK^3 cokurtosis matrix:

$$\begin{aligned}
\mu_F &= E[F] \\
S &= E\left[(F-\mu_F)(F-\mu_F)'\right] \\
G &= E\left[(F-\mu_F)(F-\mu_F)'\otimes(F-\mu_F)'\right] \\
P &= E\left[(F-\mu_F)(F-\mu_F)'\otimes(F-\mu_F)'\otimes(F-\mu_F)'\right]
\end{aligned} \tag{11}$$

We rewrite the comoment matrices *Σ, Φ* and *Ψ* as the sum of the comoment of the return explained by the factor (i.e. Bf_t) and a residual matrix denoted by *Δ, Ω* and *Y*:

$$\begin{aligned} \Sigma &= BSB + \Delta, \\ \Phi &= BG(B' \otimes B') + \Omega, \\ \Psi &= BG(B' \otimes B' \otimes B') + \Upsilon. \end{aligned} \quad (12)$$

Because of the assumption that the unexplained asset return variation e_t is independent of the factors, *Δ* is a diagonal matrix with *i*th diagonal element equal to the variance of the *i*th error term and *Ω* is a NxN^2 matrix of zeros except for the *i,j* elements where *j=(i-1)N+I*, which is corresponding to the expected third moment of the idiosyncratic factors.

The definition of *Y* is slightly more complex. Like the other residual matrices, it consists mostly of zeros, except for the cokurtosis elements $\psi_{i,j,k,l}$ corresponding to the decomposition of: the kurtosis of one asset, the cokurtosis between two assets and the cokurtosis between 3 assets (see Appendix). For the kurtosis of one asset (i.e. *i = j = k = l*), the corresponding element in *Y* should be:

$$6b_i^{'}Sb_i\ E\left[e_{it}^2\right] + E\left[e_{it}^4\right]. \quad (13)$$

For the cokurtosis between two assets when *(i=j=k)* and *l≠i*,[3] the corresponding element in *Y* should be:

$$3b_i^{'}Sb_l\ E\left[e_{it}^2\right]. \quad (14)$$

When *(i = j)* ≠ (*k = l)*,[4] the corresponding element in *Y* should be:

$$b_i^{'}Sb_i\ E\left[e_{kt}^2\right] + b_k^{'}Sb_k\ E\left[e_{it}^2\right] + E\left[e_{it}^2\right]E\left[e_{kt}^2\right]. \quad (15)$$

Finally, for the cokurtosis between 3 assets, i.e. *i = j and k≠i, k≠l, l≠l,*[5] the corresponding element in *Y* should be:

$$b_k^{'}Sb_l\ E\left[e_{it}^2\right]. \quad (16)$$

See Appendix for the proof.[6]

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[3] And analogously for *(i=j=l)* and *k≠l*, or *(j=k=l)* and *i≠k*.
[4] And analogously for *(i = l)* ≠ (*j = k)* and *(i = k)* ≠ (*j = l).*
[5] And analogously for similar combinations.
[6] For the special case of one factor (*K=1*), these decompositions were first made by Martellini and Ziemann (2010). Our general results correspond with theirs when *K=1*, except for *i = j = k = l*; for which they omitted the term $6b_i^{'}Sb_i\ E\left[e_{it}^2\right]$ in their equation (26).

Empirical illustration

The proposed methodology has important applications in the design of optimal asset allocation portfolios. We illustrate this for a realistic universe of four equity benchmarks (Europe, North America, Pacific and Emerging Markets), nine bond indices (corporate developed high yield index in EUR and USD, corporate developed investment grade index in EUR and USD, corporate emerging investment grade in USD, sovereign developed investment grade in USD, EUR, JPY and sovereign developed and emerging in USD) and five commodity indices (agriculture, energy, industrial metals, livestock and precious metals).

The cumulative return evolution of each of the asset classes over the period 1999-2012 is shown in Figure 1. Besides the differences in volatility and return over the period, the graph clearly shows the diversification potential across the different investment universes.

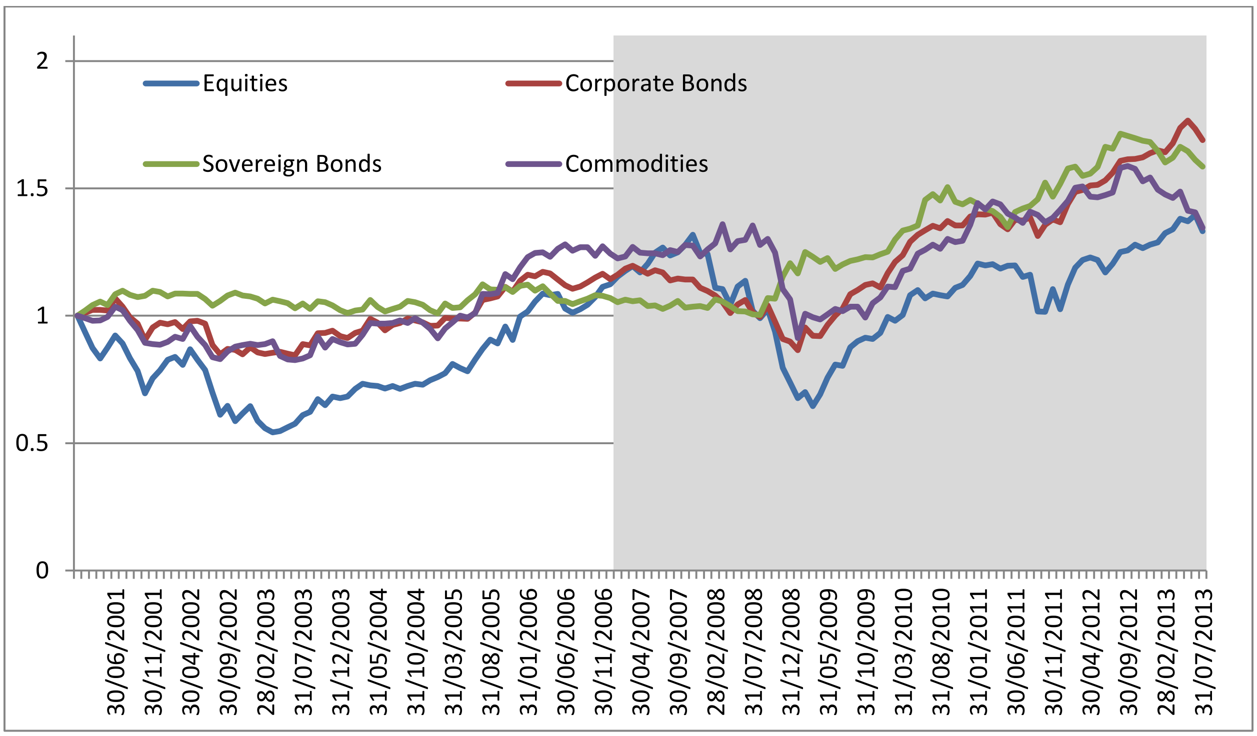

Figure 1 Monthly cumulative level of equal-weighted equity, equal-weighted sovereign bonds, equal-weighted corporate bonds and equal-weighted commodities over the period February 2001-July 2013 (in EUR). The grey area indicates the out-of-sample evaluation period used.

The shaded area in Figure 1 corresponds to the out-of-sample evaluation period used to compare the different portfolio allocation methodologies. Table 1 summarizes the performance of the different assets over this period (expressed in EUR).

The broad picture of Table 1 is that, over the period, the bond asset class clearly outperformed the equity and commodities asset classes, but there is substantial variation in return and risk of the components of the same asset class.

In equities, the North American benchmark yielded the highest annualized return (4.35%), the lowest annualized volatility (16.11%) and the most positive skewness (-0.428), while European markets showed the lowest annualized returns (0.11%) and the most negative skewness (-0.736). In the bond market, the impact of the issuer is as intuitively expected. Corporate HY bonds exhibit a higher return and risk than corporate IG and sovereign developed. Currency also has a substantial impact that seems to interact with the effect of the issuer. Finally, except for precious metals, commodities had negative returns over the period.

Table 1 Monthly returns analysis of single-asset benchmarks over the period January 2007 – July 2013 (in EUR).

	Annualized return	Annualized standard deviation	Skewness	Excess kurtosis	95% Historical VaR	Max Drawdown
Equity Europe	0.11%	18.05%	-0.736	1.162	-9.28%	-53.92%
Equity North America	4.35%	16.11%	-0.428	0.173	-7.67%	-46.46%
Equity Emerging Markets	3.00%	20.46%	-0.639	1.275	-10.01%	-55.83%
Equity Pacific	1.07%	15.10%	-0.470	-0.069	-7.51%	-45.99%
Corp Developed HY, EUR	7.34%	14.32%	-1.128	7.808	-6.65%	-36.06%
Corp Developed HY, USD	8.22%	13.18%	1.955	8.827	-2.44%	-26.78%
Corp Developed IG, EUR	5.23%	11.51%	0.458	-0.226	-4.52%	-15.00%
Corp Emerging IG, USD	6.79%	11.32%	0.740	3.178	-3.81%	-21.39%
Sov Developed IG, EUR	5.28%	5.39%	0.103	2.657	-1.96%	-8.49%
Sov Developed IG, JPY	4.95%	16.48%	1.808	7.219	-3.86%	-25.47%
Sov Developed IG, USD	6.24%	13.35%	0.932	1.700	-4.51%	-15.02%
Sov Emerging IG,USD	7.47%	10.22%	0.560	0.803	-3.65%	-12.70%
Precious metals	9.68%	19.75%	0.089	0.209	-8.22%	-33.86%
Energy	-2.72%	28.70%	-1.387	4.194	-15.64%	-69.36%
Industry Metals	-5.48%	24.23%	-0.479	1.353	-12.33%	-62.46%
Agriculture	-0.29%	23.72%	0.274	0.081	-10.44%	-45.44%
Livestock	-8.45%	17.58%	0.133	0.541	-8.65%	-52.83%

Table 2 compares the portfolio methodologies, which either aim to minimize variance, or to maximize the CRRA expected utility with risk aversion parameter γ (5 and 10), under the risk diversification constraint that all three major asset classes contribute equally to portfolio variance or to the 95% portfolio expected shortfall. Portfolios are fully invested and short sales positions are not allowed. This leads to six constrained portfolio objectives. Each objective depends on the estimated comoments, for which we evaluate three options: the sample estimator, the single factor model based estimate and a 3-factor approach. In total we have thus 18 portfolio allocation schemes. The portfolios are optimized using the Differential Evolution algorithm as explained in Ardia et al (2011). The comoments are estimated using weekly returns in local currency and a rolling sample of six years. The factor exposures are estimated by the ordinary least squares estimator and population moments are estimated by their sample moments.

The standard approach is in the first row of Table 2. It consists in minimizing the portfolio variance under the equal variance contribution constraint. Comparing this method with the other 17 portfolio allocation methodologies, the overall picture appears to be that the following three steps tend to increase the out-of-sample return and reduce the portfolio risk:

- first, the estimation of comoments under a multifactor specification;
- second, the inclusion of higher order comoments in the objective function by using a maximum CRRA expected utility objective instead of a minimum variance objective; and
- third, controlling diversification through an equal expected shortfall contribution constraint instead of an equal variance contribution constraint.

These effects are expected, since asset returns are non-normally distributed, as shown by the skewness and excess kurtosis in Table 1.

For our sample, the best risk-adjusted portfolio performance is observed for the maximum expected utility objective with risk aversion parameter γ =10 and diversification imposed through an equal contribution to the expected shortfall constraint, whereby the comoments are estimated under the multifactor approach. This scenario shows an annualized return of 9.47%, an annualized volatility of 9.35%, positive skewness and a 95% portfolio VaR of 2.25%. Its maximum drawdown (8.06%) is less than the maximum drawdown on the least risky asset of the universe (Sov Developed IG, EUR: 8.46% maximum drawdown) at a substantially higher return (9.47% instead of 5.28%), and, importantly, because of the diversification constraint, it has a much smaller idiosyncratic risk.

Table 2 Monthly returns analysis of minimum variance/maximum expected utility strategies under equal variance/ES constraint over the period January 2007 – July 2013 (in EUR).

Objective/ Risk Contribution	Estimator	Ann. return	Annualized standard deviation	Skewness	Excess kurtosis	95% Historical VaR	Max Drawdown
Min Variance/	Sample	1.02%	10.69%	-1.60	9.02	-5.60%	-25.52%
Variance	Single factor	3.02%	7.54%	0.49	1.67	-2.89%	-8.99%
	Multifactor	3.48%	7.84%	0.71	1.73	-2.83%	-18.56%
Min Variance/	Sample	5.08%	8.46%	0.17	0.90	-3.38%	-16.45%
E.Shortfall	Single factor	5.54%	7.90%	0.40	0.55	-2.96%	-7.86%
	Multifactor	4.70%	8.35%	0.33	0.52	-3.27%	-12.40%
Max E.Utility/	Sample	3.36%	8.32%	0.06	2.98	-3.27%	-9.89%
Variance	Single factor	5.85%	8.57%	0.64	0.70	-3.04%	-10.26%
(γ=5)	Multifactor	4.23%	8.73%	1.06	3.41	-3.44%	-13.27%
Max E.Utility/	Sample	5.47%	7.07%	0.27	0.77	-2.68%	-9.17%
E.Shortfall	Single factor	4.43%	9.59%	-0.88	6.75	-4.40%	-18.24%
(γ=5)	Multifactor	5.21%	9.67%	-0.98	6.88	-4.43%	-13.31%
Max E.Utility/	Sample	3.64%	8.03%	0.38	0.29	-3.19%	-11.65%
Variance	Single factor	1.80%	7.97%	0.01	0.57	-3.55%	-11.67%
(γ=10)	Multifactor	4.53%	8.90%	0.27	-0.18	-3.59%	-11.87%
Max E.Utility/	Sample	4.82%	9.05%	-0.05	0.38	-3.86%	-11.99%
E.Shortfall	Single factor	4.49%	8.25%	0.26	0.91	-3.28%	-10.27%
(γ=10)	**Multifactor**	**9.47%**	**9.35%**	**1.43**	**3.32**	**-2.25%**	**-8.06%**

Finally, we report in Figure 2 the monthly levels of the benchmark minimum variance portfolio using the sample estimator, and compare with the higher order comoment based maximum expected utility (γ =10) portfolios using the three types of estimator. Clearly, the objective function matters and over the period, the maximum expected utility objective with expected shortfall as risk measure leads to significantly better performance than the standard approach based on the variance as risk measure. As mentioned above, in the sample, the choice of the estimator seems to matter, as the portfolio using the comoments estimated under the three-factor model has a substantially higher performance over the period.

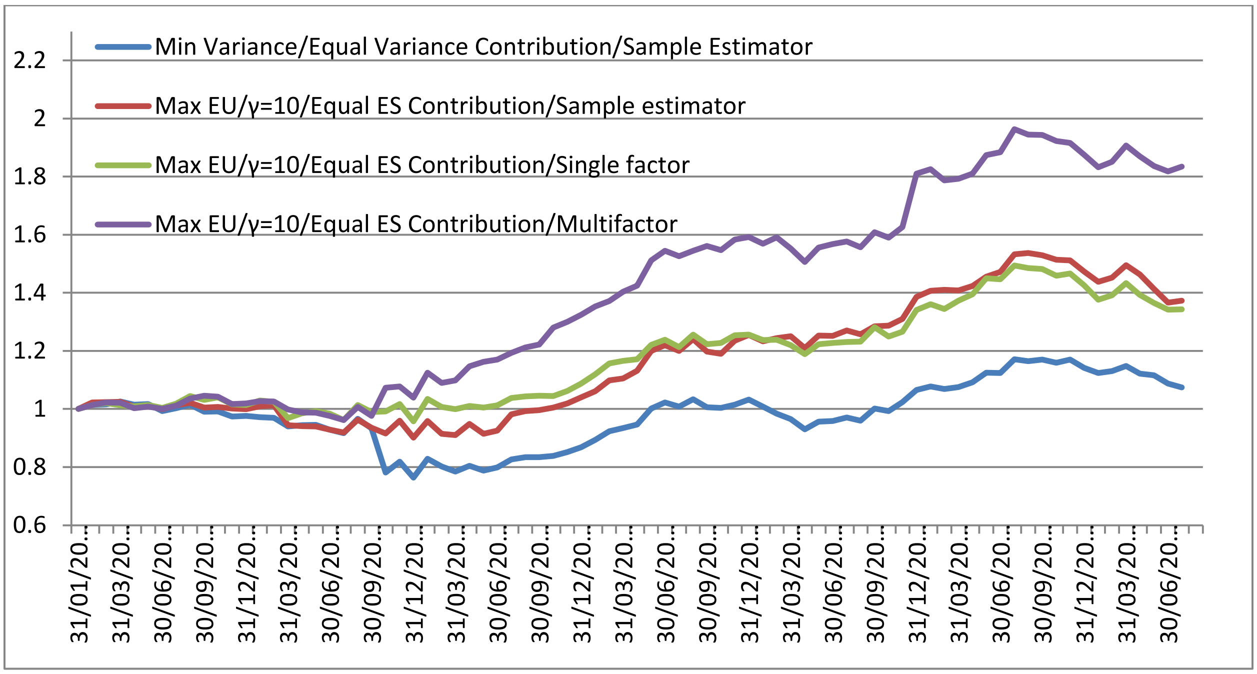

Figure 2 Monthly cumulative level in EUR of the weekly rebalanced international asset allocation portfolio that either minimized variance or maximizes the expected utility with γ=10, under the diversification constraint of equal variance or equal ES contribution of each asset class. The comoments are estimated using the sample approach, single factor model or three-factor model.

Conclusion

We propose a novel methodology to account for non-normality in a multifactor world. The multifactor assumption allows for parsimoniously estimating the higher order comoments. Parsimony is achieved by striking a balance between generality (3 factors) and the number of parameters to estimate. It extends the single factor approach in Martellini and Ziemann (2010). This is the key contribution of this article.

Additionally, we conduct an explorative empirical analysis in order to assess the potential gains of including higher order comoments in the asset allocation decision. For this, we apply comoment estimates in an out-of-sample portfolio analysis for a heterogeneous universe of 19 equity, bonds and commodity indices over the period January 2007-July 2013.

Our analysis seems to suggest that that the following three actions may increase the out-of-sample return and reduce the portfolio risk: (i) imposing more structure on the estimates (moving from the sample estimator to the factor model based estimates); (ii) switching from an equal variance contribution constraint to an equal expected shortfall constraint; (iii) switching from a minimum variance objective to a CRRA expected utility objective.

We believe that our findings are promising but require more research in terms of sensitivity to the sample studied, turnover analysis, risk budgets, the treatment of currency effects, the handling of outliers and, importantly, the choice of factors.

In this study we used the principal components of the asset returns as the driver of the dependence across asset returns. We are currently working on applications with observed factors and developing a test for the correct specification of the number of factors required to explain the return dependence.

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APPENDIX PROOFS OF DECOMPOSITION HIGHER ORDER COMOMENTS UNDER FACTOR MODEL SPECIFICATION

For notational convenience and without loss of generality, we assume here the returns and factors to be de-meaned such that $r_{i,t} = b_i' f_t + e_{it}$. Because we assume independence between the factors and residual terms, we have that, for $j \neq k$, and powers p and q:

$$E\left[\left(b_j' f_t\right)^p e_{kt}^q\right] = E\left[\left(b_j' f_t\right)^p\right] E\left[e_{kt}^q\right].$$

The independence across residual terms implies further that, for $j \neq k$:

$$E\left[e_{jt}^p e_{kt}^q\right] = E\left[e_{jt}^p\right] E\left[e_{kt}^q\right].$$

These independence assumptions allow us to simplify greatly the higher order comoments, as we show next.

We first consider the elements in the coskewness matrix $\Phi = BG(B' \otimes B') + \Omega$. We have that for the skewness of asset i:

$$\Phi_{i,i,i} = E\left[\left(b_i' f_t + e_{it}\right)^3\right]$$

$$= E\left[\left(b_i' f_t\right)^3 + 3\left(b_i' f_t\right)^2 e_{it} + 3\left(b_i' f_t\right) e_{it}^2 + e_{it}^3\right]$$

$$= b_i' G\left(b_i \otimes b_i\right) + E\left[e_{it}^3\right].$$

Hence the residual element in Ω is $E\left[e_{it}^3\right]$. All other elements in Ω are 0, since the coskewness only comes from the common factor exposures:

$$\Phi_{i,i,j} = E\left[\left(b_i' f_t + e_{it}\right)^2\left(b_j' f_t + e_{jt}\right)\right]$$

$$= E\left[\left(b_i' f_t\right)^2\left(b_j' f_t\right) + \left(b_i' f_t\right)^2 e_{jt} + 2\left(b_i' f_t\right) e_{it}\left(b_j' f_t\right) + 2\left(b_i' f_t\right) e_{it} e_{jt} + e_{it}^2\left(b_j' f_t\right) + e_{it}^2 e_{jt}\right]$$

$$= b_i' G\left(b_i \otimes b_j\right),$$

for $i \neq j$, and, for $i \neq j$, $j \neq k$, and $i \neq k$:

$$\Phi_{i,j,k} = E\left[\left(b_i' f_t + e_{it}\right)\left(b_j' f_t + e_{jt}\right)\left(b_k' f_t + e_{kt}\right)\right]$$

$$= b_i' G\left(b_j \otimes b_k\right).$$

We now consider the cokurtosis matrix $\Psi = BG(B'\otimes B'\otimes B') + \Upsilon$. For the kurtosis of asset *i*:

$\Psi_{i,i,i,i} = E[\,(b_i' f_t + e_{it})^4]$

$= \ E\big[\,(b_i' f_t)^4 + 4(b_i' f_t)^3 e_{it} + 6(b_i' f_t)^2 e_{it}^2 + 4(b_i' f_t) e_{it}^3 + e_{it}^4\big]$

$= \ b_i' G(b_i \otimes b_i \otimes b_i\,) + 6 b_i' S b_i E\big[e_{it}^2\big] + E\big[e_{it}^4\big].$

The result in (13) then follows.

Then for the cokurtosis element, where *(i=j=k)* and *l≠i*:

$\Psi_{i,i,i,l} = E[\,(b_i' f_t + e_{it})^3 (b_l' f_t + e_{lt})]$

$= \ E\big[\,\big((b_i' f_t)^3 + 3(b_i' f_t)^2 e_{it} + 3(b_i' f_t) e_{it}^2 + e_{it}^3\big)\,(b_l' f_t + e_{lt})\big]$

$= \ b_i' G\big(b_i \otimes b_i \otimes b_j\,\big) + 3E\big[(b_i' f_t)(b_l' f_t) e_{it}^2\big]$

$= \ b_i' G\big(b_i \otimes b_i \otimes b_j\,\big) + 3 b_i' S b_l E\big[e_{it}^2\big].$

This proves the result in (14). We obtain (15) from:

$\Psi_{i,i,k,k} = E[\,(b_i' f_t + e_{it})^2 (b_k' f_t + e_{kt})^2]$

$= \ E\big[\,\big((b_i' f_t)^2 + 2(b_i' f_t) e_{it} + e_{it}^2\big)\,\big((b_k' f_t)^2 + 2(b_k' f_t) e_{kt} + e_{kt}^2\big)\big]$

$= \ b_i' G(b_i \otimes b_k \otimes b_k\,) + b_i' S b_i\ \ E\big[e_{kt}^2\big] + b_k' S b_k\ E\big[e_{it}^2\big] + E\big[e_{it}^2\big] E\big[e_{kt}^2\big].$

Finally, (16) is obtained using:

$\Psi_{i,i,k,l} = E[\,(b_i' f_t + e_{it})^2 (b_k' f_t + e_{kt})(b_l' f_t + e_{lt})]$

$= \ E\big[\,\big((b_i' f_t)^2 + 2(b_i' f_t) e_{it} + e_{it}^2\big)\,(b_k' f_t + e_{kt})(b_l' f_t + e_{lt})\big]$

$= \ b_i' G(b_i \otimes b_k \otimes b_l\,) + b_k' S b_l\ \ E\big[e_{it}^2\big],$

where $i \neq k$, $k \neq l$, and $i \neq l$.

For all other elements that have no common index:

$\Psi_{i,j,k,l} = E\big[\,(b_i' f_t + e_{it})\big(b_j' f_t + e_{jt}\big)(b_k' f_t + e_{kt})(b_l' f_t + e_{lt})\big]$

$= \ b_i' G\big(b_j \otimes b_k \otimes b_l\,\big).$

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